

COMMUNICATIONS

Mathematics

ON LEBESGUE CONSTANTS OF VILENKIN SYSTEMS

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In the paper some properties of Lebesgue constants $\{L_n(W)\}_{n=1}^\infty$ of Vilenkin system are investigated. Non almost convergence property for the sequence $\left\{ \frac{L_n(W)}{\log_2 n} \right\}_{n=2}^\infty$ is obtained.

MSC2010: Primary 42C10, Secondary 42B08.

Keywords: Vilenkin system, Lebesgue constants, almost convergence.

Introduction. Let $\Psi = \{\psi_k(x)\}_{k=1}^\infty$ be an orthonormal system of functions defined on $[a, b]$. The Lebesgue constants of Ψ is defined as follows:

$$L_n(\Psi, x) := \int_a^b \left| \sum_{k=1}^n \psi_k(x) \overline{\psi_k(t)} \right| dt \quad (\text{here } \bar{x} \text{ means the complex conjugate of } x).$$

If this functions are independent on x , then they are called Lebesgue constants $\{L_n(\Psi)\}_{n=1}^\infty$ of system Ψ . Recall the definition of Vilenkin systems. Consider an arbitrary sequence of natural numbers $P \equiv \{p_1, p_2, \dots, p_k, \dots\}$, where $p_j \geq 2$ for all $j \in \mathbb{N}$.

Let denote $m_0 = 1$, $m_k = \prod_{j=1}^k p_j$ ($p_j \geq 2$).

It is easy to see that for each $x \in [0, 1)$ and for each $n \in \mathbb{N}$ there exist integers $x_j, \alpha_j \in \{0, 1, \dots, p_j - 1\}$ (in the case $x = \frac{l}{m_k}$, $l \in \mathbb{N}$, $0 \leq l \leq m_k - 1$, we take $x_j = 0$ for all $j > k$), so that $n = \sum_{j=1}^\infty \alpha_j m_{j-1}$ and $x = \sum_{j=1}^\infty \frac{x_j}{m_j}$ (P -adic expansions).

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Vilenkin or multiplicative system with respect to the sequence P is defined as follows:

$$W_0(x) \equiv 1; W_n(x) = \exp \left(2\pi i \sum_{j=1}^k \alpha_j \frac{x_j}{p_j} \right).$$

Obviously the n th function can be represented by

$$W_n(x) = \prod_{j=1}^k (W_{m_{j-1}}(x))^{\alpha_j}.$$

Note that systems corresponding to distinct sequences $P = \{p_k\}$, are different, in particular if $P \equiv \{2, 2, \dots, 2, \dots\}$ the Vilenkin system coincides with the Walsh one (see [1]). The theory of these systems was developed by N. Ya. Vilenkin in 1946 (see [2, 3]).

The Lebesgue constants of Vilenkin system have the form

$$L_n(W) = \int_0^1 |D_n(t)| dt, \quad (1)$$

where $D_n(t) = \sum_{k=0}^{n-1} W_k(t)$ is the n th Dirichlet kernel of Vilenkin system.

It is known [4] that $\lim_{n \rightarrow \infty} \frac{L_n(T)}{\ln n} = \frac{4}{\pi}$, where T is the trygonometric system. Note that, in contrast to this, for the Vilenkin system it was proved [3] that

$$0 = \liminf_{n \rightarrow \infty} \frac{L_n(W)}{\log_2 n} < \limsup_{n \rightarrow \infty} \frac{L_n(W)}{\log_2 n} < \infty. \quad (2)$$

Recall that a bounded sequence $\{x_n\}_{n=1}^{\infty}$ is called almost convergent, if for some $a \in \mathbb{R}$ we have $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=m+1}^{m+n} x_k = a$ uniformly by m . Denote

$$q(x_n) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=m+1}^{m+n} x_k \text{ and } p(x_n) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=m+1}^{m+n} x_k.$$

This limits exist for every bounded sequence, and obviously the almost convergence of a sequence $\{x_n\}_{n=1}^{\infty}$ is equivalent to the condition $q(x_n) = p(x_n)$ (see [5, 6]).

In this paper we prove the following

Theorem 1. For any Vilenkin system the following equivalencies are true:

1. $q \left(\frac{L_n(W)}{\log_2 n} \right) = 0 \left(= \liminf_{n \rightarrow \infty} \frac{L_n(W)}{\log_2 n} \right);$
2. $p \left(\frac{L_n(W)}{\log_2 n} \right) = \limsup_{n \rightarrow \infty} \frac{L_n(W)}{\log_2 n} < \infty.$

From this Theorem as a direct consequence we obtain that the sequence $\left\{ \frac{L_n(W)}{\log_2 n} \right\}_{n=2}^{\infty}$ is not almost convergent. Note that the analogues result for the Walsh system is formulated in [6].

Auxiliary Propositions. We will use the following properties of Vilenkin system

$$L_{m_k}(W) = 1, \quad k = 0, 1, \dots, \quad (3)$$

$$W_{lm_k+\beta}(x) = W_{lm_k}(x)W_\beta(x) \text{ if } \beta < m_k \quad (k, l, \beta \in \mathbb{N}). \quad (4)$$

From (4) we get

$$\begin{aligned} D_{m_k+r}(t) &\equiv \sum_{j=0}^{m_k-1} W_j(t) + W_{m_k}(t) \sum_{j=0}^{r-1} W_j(t) \equiv \\ &\equiv D_{m_k}(t) + W_{m_k}(t)D_r(t) \text{ for all } 1 < r \leq m_k. \end{aligned} \quad (5)$$

Proof of Main Result. Let us begin with a proof of first equation.

We put

$$l_n := \frac{L_n(W)}{\log_2 n}, \quad \check{l}_n = \inf_{m \in \mathbb{N}} \frac{1}{n} \sum_{k=m+1}^{m+n} l_k \quad \text{and} \quad \hat{l}_n = \sup_{m \in \mathbb{N}} \frac{1}{n} \sum_{k=m+1}^{m+n} l_k.$$

Let $\varepsilon > 0$ be an arbitrary positive number and $n \in \mathbb{N}$. Obviously there exists $k \in \mathbb{N}$ depending on n such that

$$m_k > n \quad \text{and} \quad \frac{1}{k} \left(\frac{1}{n} \sum_{r=1}^n (1 + L_r) \right) < \varepsilon. \quad (6)$$

From (1), (3) and (5) we obtain

$$L_{m_k+r} \leq 1 + L_r \quad \text{for all } 1 \leq r \leq n.$$

Combining this with (6) and taking into consideration a relation $m_k \geq 2^k$, we get

$$\frac{1}{n} \sum_{j=m_k+1}^{m_k+n} l_j \leq \frac{1}{k} \left(\frac{1}{n} \sum_{r=1}^n (1 + L_r) \right) < \varepsilon.$$

Therefore we have $0 \leq \check{l}_n < \varepsilon$ for any natural number n , and eventually since ε is arbitrary we get the first equation $q(l_n) = 0$.

Next we prove that $\lim_{r \rightarrow \infty} \hat{l}_r \geq c$. We put

$$c = \limsup_{n \rightarrow \infty} l_n, \quad c_1 = \sup_{n \in \mathbb{N}} l_n. \quad (7)$$

Let r be any natural number and $\varepsilon > 0$. We fix $k_0 \in \mathbb{N}$ such that $m_{k_0} \geq r$, then we take $n_0 \in \mathbb{N}$ ($n_0 > m_{k_0}$), so that

$$l_{n_0} > c - \frac{\varepsilon}{3}, \quad \frac{\log_2 m_{k_0}}{\log_2(n_0 - m_{k_0})} < \frac{\varepsilon}{6c_1} \quad \text{and} \quad \frac{\log_2 n_0}{\log_2(n_0 + m_{k_0})} > 1 - \frac{\varepsilon}{3(c+1)}. \quad (8)$$

The P -adic expansion of n_0 has the form $n_0 = \sum_{j=0}^k \alpha_j m_j$. Denote

$$n'_0 := \sum_{j=k_0}^k \alpha_j m_j. \quad (9)$$

Let $n \in [n'_0, n'_0 + m_{k_0})$. By the same argument as in (5) we get

$$D_n(t) \equiv D_{n'_0}(t) + W_{n'_0}(t)D_{n-n'_0}(t).$$

Thus, from (1) and reverse triangle inequality, we obtain

$$L_n \geq L_{n'_0} - L_{n-n'_0} \quad \text{and} \quad L_{n'_0} \geq L_{n_0} - L_{n-n'_0}.$$

Hence (see also (7)),

$$L_n \geq L_{n_0} - 2L_{n-n'_0} \geq L_{n_0} - 2c_1 \log_2 m_{k_0}.$$

From this and (8), (9) we have

$$l_n \geq \frac{L_{n_0}}{\log_2 n_0} \left(\frac{\log_2 n_0}{\log_2 n} \right) - 2 \frac{c_1 \log_2 m_{k_0}}{\log_2 n} > c - \varepsilon. \quad (10)$$

From (10) we get

$$\frac{1}{r} \sum_{n=n'_0+1}^{n'_0+r} l_n > c - \varepsilon,$$

which implies $\hat{l}_r \geq c - \varepsilon$, and from arbitrariness of r and ε we obtain that $\lim_{r \rightarrow \infty} \hat{l}_r \geq c$.

It remains to show that $\lim_{r \rightarrow \infty} \hat{l}_r \leq c$.

Again we take $\varepsilon > 0$ to be an arbitrary number, then we choose natural numbers k_0 and $k'_0 > k_0$ such that

$$l_k < c + \frac{\varepsilon}{2} \quad \text{for all} \quad k \geq k_0 \quad \text{and} \quad \frac{1}{k'_0} \sum_{k=1}^{k_0} l_k < \frac{\varepsilon}{2}. \quad (11)$$

If $m < k_0$ or all $r \geq k'_0$, we have

$$\frac{1}{r} \sum_{k=m+1}^{m+r} l_k = \frac{1}{r} \sum_{k=m+1}^{k_0} l_k + \frac{1}{r} \sum_{k=k_0+1}^{m+r} l_k. \quad (12)$$

From (11) and (12) we get

$$\frac{1}{r} \sum_{k=m+1}^{m+r} l_k \leq c + \varepsilon \quad \text{for all} \quad r \geq k'_0.$$

If $m \geq k_0$, then from (11) for all $r \in \mathbb{N}$ we obtain

$$\frac{1}{r} \sum_{k=m+1}^{m+r} l_k \leq c + \frac{\varepsilon}{2}$$

Hence $\hat{l}_r \leq c + \varepsilon$ for all $r \geq k'_0$ and eventually we get $\lim_{r \rightarrow \infty} \hat{l}_r \leq c$. \square

Received 08.02.2016

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