PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2015, № 1, p. 52–60

I n f o r m a t i c s

ON NON-CLASSICAL THEORY OF COMPUTABILITY

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Definition of arithmetical functions with indeterminate values of arguments is given. Notions of computability, strong computability and λ-definability for such functions are introduced. Monotonicity and computability of every λ-definable arithmetical function with indeterminate values of arguments is proved. It is proved that every computable, naturally extended arithmetical function with indeterminate values of arguments is λ-definable. It is also proved that there exist strong computable, monotonic arithmetical functions with indeterminate values of arguments, which are not λ-definable. The δ-redex problem for strong computable, monotonic arithmetical functions with indeterminate values of arguments is defined. It is proved that there exist strong computable, λ-definable arithmetical functions with indeterminate values of arguments, for which the δ-redex problem is unsolvable.

MSC2010: 68Q01; 68Q05.

***Keywords*:** arithmetical function, indeterminate value of argument, computability, strong computability, λ-definability, β-redex, δ-redex.

Introduction. The paper is devoted to arithmetical functions with indeterminate values of arguments. These functions are defined on the partially ordered set $M = N \cup \{\perp\}$, where N is the set of natural numbers, $\perp$ is the element, which corresponds to indeterminate value. Each element of M is comparable with itself and with $\perp$, which is the least element of M. The notion of monotonic function is introduced in the conventional way. A function is said to be naturally extended, if its value is $\perp$ whenever the value of at least one of the arguments is $\perp$. These kind of functions have been considered in [1]. In this paper the research is presented, the start of which was given in [2].

Notions of computability and strong computability for arithmetical functions with indeterminate values of arguments are introduced. The class of partial recursive functions with indeterminate values of arguments is defined. It is proved that the class of computable, naturally extended arithmetical functions with indeterminate

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values of arguments and the class of partial recursive functions with indeterminate values of arguments are the same. The notion of primitive recursive function with indeterminate values of arguments is introduced. It is proved that naturally extended arithmetical functions with indeterminate values of arguments, for which the domain of definition is finite, are primitive recursive.

Notion of λ-definability for arithmetical functions with indeterminate values of arguments is introduced. It is proved that every λ-definable arithmetical function with indeterminate values of arguments is monotonic and computable. It is proved, also, that every computable, naturally extended arithmetical function with indeterminate values of arguments is λ-definable. It is proved that there exist strong computable, monotonic, not naturally extended, arithmetical functions with indeterminate values of arguments, which are not λ-definable. It is also proved that there exist strong computable, monotonic, not naturally extended arithmetical functions with indeterminate values of arguments, which are λ-definable.

The δ-redex problem for strong computable, monotonic arithmetical functions with indeterminate values of arguments is defined. An expression $\varphi(\upsilon_1,\ldots,\upsilon_k)$, where $\varphi : M^k \to M, k \geq 1$, is a strong computable, monotonic arithmetical function with indeterminate values of arguments, υ_i is from M or is a variable, $i = 1,\ldots,k$, is called a δ-redex, if the value of the expression $\varphi(\upsilon_1,\ldots,\upsilon_k)$ is the same for any value of the variables. It is proved that there exist strong computable, λ-definable arithmetical functions with indeterminate values of arguments for which the δ-redex problem is unsolvable.

Arithmetical Functions with Indeterminate Values of Arguments. Let $M = N \cup \{\perp\}$, where $N = \{0,1,2,\ldots\}$ is the set of natural numbers, $\perp$ is the element corresponding to the indeterminate value. Let us introduce the partial ordering $\subseteq$ on the set M. For every $m \in M$ we have: $\perp \subseteq m$ and $m \subseteq m$. A mapping $\varphi : M^k \to M$,where $k \geq 1$, is said to be an arithmetical function with indeterminate values of arguments.

A function $\varphi : M^k \to M,\ k \geq 1$, is said to be computable, if there exists an algorithm (Turing machine, see [3,4]), which for all $m_1,\ldots,m_k \in M$ stops with value $\varphi(m_1,\ldots,m_k)$, if $\varphi(m1,\ldots,m_k) \neq \perp$, and stops with the value $\perp$, or works infinitely, if $\varphi(m_1,\ldots,m_k) = \perp$.

A function $\varphi : M^k \to M,\ k \geq 1$, is said to be strong computable, if there exists an algorithm (Turing machine, see [3,4]), which stops with the value $\varphi(m_1,\ldots,m_k)$ for all $m_1,\ldots,m_k \in M$.

A function $\varphi : M^k \to M,\ k \geq 1$, is said to be monotonic, if $(m_1,\ldots,m_k) \subseteq (\mu_1 \ldots,\mu_k)$ implies $\varphi(m_1,\ldots,m_k) \subseteq \varphi(\mu_1 \ldots,\mu_k)$ for all $m_i,\ \mu_i \in M, i = 1,\ldots,k$. A function $\varphi : M^k \to M,\ k \geq 1$, is said to be naturally extended, if $\varphi(...,\perp,...) = \perp$. It is easy to see that every naturally extended function is monotonic.

Let $\varphi : M^k \to M,\ k \geq 1$, be a naturally extended arithmetical function with indeterminate values of arguments and

$$\mathrm{Arg}(\varphi) = \{(n_1,...,n_k) \in N^k | \varphi(n_1,...,n_k) \neq \perp\}.$$

It is easy to see that every computable, naturally extended arithmetical function with indeterminate values of arguments $\varphi : M^k \to M,\ k \geq 1$, for which $\mathrm{Arg}(\varphi) = N^k$, is a strong computable function.

It is also easy to see that every naturally extended arithmetical function with indeterminate values of arguments $\varphi : M^k \to M,\ k \geq 1$, for which the set $\mathrm{Arg}(\varphi)$ is finite, is a strong computable function.

The class of partial recursive functions with indeterminate values of arguments is defined as follows:

1. Base functions o, s, $I_{k,i}$, $1 \leq i \leq k$, $k \geq 1$ (which are the natural extensions of the classical base functions, see [3]) are partial recursive functions with indeterminate values of arguments, where for all m, $m_1, \dots, m_k \in M$ we have:

$o(m)$ equals 0, if $m \in N$, and equals $\perp$ if $m = \perp$,

$s(m)$ equals $m+1$, if $m \in N$, and equals $\perp$ if $m = \perp$,

$I_{k,i}(m_1, \dots, m_k)$ equals m_i, if $m_1, \dots, m_k \in N$, and equals $\perp$ otherwise.

2. If $h : M^r \to M$ and $g_1, \dots, g_r : M^k \to M, r, k \geq 1$, are partial recursive functions with indeterminate values of arguments, then so is the function $\varphi : M^k \to M$, defined by the composition, where for all $m_1, \dots, m_k \in M$ we have:

$$\varphi(m_1, \dots, m_k) = h(g_1(m_1, \dots, m_k), \dots, g_r(m_1, \dots, m_k)).$$

3. Consider two following cases:

3a. If $k = 1$.
If $m \in M$ and $h : M^2 \to M$ is a partial recursive function with indeterminate values of arguments, then so is the function $\varphi : M \to M$ defined by primitive recursion, where

$\varphi(\perp) = \perp$,

$\varphi(0) = m$,

$\varphi(n+1) = h(n, \varphi(n))$, where $n \in N$.

3b. If $k > 1$.
If $g : M^{k-1} \to M$ and $h : M^{k+1} \to M, k \geq 2$, are partial recursive functions with indeterminate values of arguments, then so is the function $\varphi : M^k \to M$ defined by primitive recursion, where for all $m_1, \dots, m_{k-1} \in M$ we have:

$\varphi(m_1, \dots, m_{k-1}, \perp) = \perp$,

$\varphi(m_1, \dots, m_{k-1}, 0) = g(m_1, \dots, m_{k-1})$,

$\varphi(m_1, \dots, m_{k-1}, n+1) = h(m_1, \dots, m_{k-1}, n, \varphi(m_1, \dots, m_{k-1}, n))$, where $n \in N$.

4. If $g : M^{k+1} \to M,\ k \geq 1$, is partial recursive function with indeterminate values of arguments, then so is the function $\varphi : M^k \to M$ defined by minimization, where for all $m_1, \dots, m_k \in M$ we have:

$$\varphi(m_1, \dots, m_k) = \begin{cases} n \in N, & \text{if } g(m_1, \dots, m_k, n) = 0 \text{ and for all} \\ & n' \in N, n' < n, g(m_1, \dots, m_k, n') \neq 0 \\ & \text{and } g(m_1, \dots, m_k, n') \neq \perp, \\ \perp, & \text{otherwise.} \end{cases}$$

Theorem 1. The class of partial recursive functions with indeterminate values of arguments and the class of computable, naturally extended arithmetical functions with indeterminate values of arguments are the same.

P r o o f . It is easy to see that every partial recursive function with indeterminate values of arguments is a computable, naturally extended arithmetical function with indeterminate values of arguments. Let $\varphi : M^k \to M, k \geq 1$, be a computable, naturally extended arithmetical function with indeterminate values of arguments. Let's show that φ is a partial recursive function with indeterminate values of arguments. Two cases are considered:

Case 1. $\mathrm{Arg}(\varphi) = \emptyset$.
In this case φ is a completely undefined function, i.e. $\varphi(m_1, \ldots, m_k) = \perp$ for all $m_1, \ldots, m_k \in M$, and it is easy to show that φ is a partial recursive function.

Case 2. $\mathrm{Arg}(\varphi) \neq \emptyset$.
Let $A = Arg(\varphi)$ and $\psi : A \to N$ be the arithmetical function (in the classical sense, see [3]), such that if $(n_1, \ldots, n_k) \in A$, then $\psi(n_1, \ldots, n_k) = \varphi(n_1, \ldots, n_k)$. It is easy to see that ψ is a computable function, i.e. ψ is a partial recursive function (in the classical sense [3]). Therefore, one can obtain ψ by using classical base functions and classical operations of composition, primitive recursion and minimization. Therefore, one can obtain φ (the same way), by using naturally extended classical base functions and operations of composition, primitive recursion and minimization from the definition of partial recursive function with indeterminate values of arguments. □

The partial recursive functions with indeterminate values of arguments, which are obtained by using only two kinds of operations: composition and primitive recursion will be called primitive recursive functions with indeterminate values of arguments. It is easy to see that such functions will be strong computable. It is obvious that there exists a strong computable, partial recursive function with indeterminate values of arguments, which is not primitive recursive, for example the Ackerman function A (see [3]), for which $A(\perp) = \perp$.

T h e o r e m 2 . Every naturally extended arithmetical function with indeterminate values of arguments $\varphi : M^k \to M, k \geq 1$, for which $\mathrm{Arg}(\varphi)$ is a finite set, is a primitive recursive function with indeterminate values of arguments.

P r o o f . The same cases are considered:

Case 1. $\mathrm{Arg}(\varphi) = \emptyset$.
In this case φ is completely undefined function, i.e. $\varphi(m_1, \ldots, m_k) = \perp$ for all $m_1, \ldots, m_k \in M$. Let's show that φ is a primitive recursive function. It is easy to see that for all $m_1, \ldots, m_k \in M$ we have $\varphi(m_1, \ldots, m_k) = \omega(I_{k,1}(m_1, \ldots, m_k))$, where for all $m \in M, \omega(m) = \perp$ and the primitive recursiveness of the function ω follows from the equalities:

$\omega(\perp) = \perp$,

$\omega(0) = \perp$,

$\omega(n+1) = I_{2,1}(n, \omega(n))$, where $n \in N$.

Case 2. $\mathrm{Arg}(\varphi) \neq \emptyset$.
Let $\mathrm{Arg}(\varphi) = \{(n_{11}, \ldots, n_{1k}), \ldots, (n_{\nu 1}, \ldots, n_{\nu k})\}$, where $n_{ij} \in N, i = 1, \ldots, \nu,\ \nu \geq 1$, $j = 1, \ldots, k,\ k \geq 1$, and $\varphi(n_{i1}, \ldots, n_{ik}) = r_i$, where $r_i \in N, i = 1, \ldots, \nu$.

It is easy to see that

$$\varphi(x_1,\dots,x_k) = g\big((r_1+1)sg'\big(|x_1-n_{11}|+\dots+|x_k-n_{1k}|\big)+\dots+ (r_v+1)sg'\big(|x_1-n_{v1}|+\dots+|x_k-n_{vk}|\big)\big),$$

where

$$sg'(x) = \begin{cases} \perp, & \text{if} \quad x=\perp, \\ 1, & \text{if} \quad x\neq\perp \text{ and } x=0, \\ 0, & \text{if} \quad x\neq\perp \text{ and } x>0. \end{cases}$$

$$g(x) = \begin{cases} \perp, & \text{if} \quad x=\perp \text{ or } x=0, \\ x-1, & \text{if} \quad x\neq\perp \text{ and } x>0. \end{cases}$$

Let's show the primitive recursiveness of the function g.

Note that such a property for the other functions in the expression of φ may be shown in the same way as their classical analogs are done. The only difference is the following: the natural extensions of the classical base functions and operations of composition and primitive recursion from the definition of a primitive recursive function with indeterminate values of arguments must be used. Observe that

$$g(x) = h(sg'(x))\cdot minus1(x),$$

where

$$minus1(x) = \begin{cases} \perp, & \text{if} \quad x=\perp, \\ 0, & \text{if} \quad x\neq\perp \text{ and } x=0, \\ x-1, & \text{if} \quad x\neq\perp \text{ and } x>0; \end{cases}$$

$$h(x) = \begin{cases} \perp, & \text{if} \quad x=\perp, \\ 1, & \text{if} \quad x\neq\perp \text{ and } x=0, \\ \perp, & \text{if} \quad x\neq\perp \text{ and } x>0. \end{cases}$$

It is easy to prove that the functions *minus*1 and h are primitive recursive functions. □

λ-Definability of Arithmetical Functions with Indeterminate Values of Arguments. Let us state some definitions and known results from [5]. Fix a countable set of variables V and define the set of terms Λ.

1. If $x\in V$, then $x\in\Lambda$;
2. if $t_1,t_2\in\Lambda$, then $(t_1t_2)\in\Lambda$;
3. if $x\in V$ and $t\in\Lambda$, then $(\lambda xt)\in\Lambda$.

We will use the abridged notation for the terms: the term $(\dots(t_1t_2)\dots t_k)$, where $t_i\in\Lambda$, $i=1,\dots,k$, $k>1$, is denoted by $t_1t_2\dots t_k$, and the term $(\lambda x_1(\lambda x_2(\dots(\lambda x_n t)\dots)))$, where $x_j\in V$, $t\in\Lambda$, $j=1,\dots,n$, $n>0$, is denoted by $\lambda x_1x_2\dots x_n.t$.

The notion of a free and bound entry of a variable in a term and the notion of a free and bound variable of a term are introduced in the traditional way. A term having no free variables is said to be closed.

Terms t_1 and t_2 are said to be congruent (which is denoted by $t_1\equiv t_2$), if one of them can be obtained from the other one by renaming of the bound variables, the congruent terms are considered identical.

A substitution of a term τ in a free entries of variable x of a term t is said to be admissible and is denoted by $t[x := \tau]$, if all the free entries of variables of the term τ remain free after the substitution. We will consider only admissible substitutions.

Let us remind the notion of the β-reduction:

$$\beta = \{((\lambda x.t)\tau, t[x := \tau]) \mid t, \tau \in \Lambda, x \in V\}.$$

A one-step β-reduction $(\to_\beta)$, β-reduction $(\to\to_\beta)$, and β-equality $(=_\beta)$ are defined in the standard way.

We remind that the term $(\lambda x.t)\tau$ is referred to as β-redex. A term not containing β-redexes is referred to as β-normal form (further, simply normal form). The set of all normal forms is denoted by NF. A term t is said to have a normal form, if there exists a term $t' \in NF$ such that $t =_\beta t'$. A term of the form $\lambda x_1 x_2 \dots x_n.xt_1t_2\dots t_k$, where $x, x_i \in V, t_j \in \Lambda, i = 1, \dots, n,\ n \geq 0, j = 1, \dots, k,\ k \geq 0$, is referred to us a head normal form. The set of all head normal forms is denoted by HNF. A term t is said to have a head normal form, if there exists a term $t' \in$HNF such that $t =_\beta t'$. It is known that NF $\subset$ HNF, but HNF $\not\subset$ NF.

We will extensively use the corollary from the CR-theorem (Church-Rosser), which says that for any term $t \in \Lambda$, the following two assertions are valid:

1. $t =_\beta t',\ t' \in NF \Rightarrow t \to\to_\beta t'$;
2. $t =_\beta t',\ t =_\beta t'',\ t', t'' \in NF \Rightarrow t' \equiv t''$.

Recall the following statement: If $t =_\beta t'$ and $t' \in$NF, then $t \to\to_\beta t'$ and $\to\to_\beta$ is the left β-reduction (i.e. the β-reduction, where, each time, the leftmost β-redex is taken). We will also use the following statement: If a term $t \in \Lambda$ does not have a head normal form, then the same holds for the term $t\tau$ for any $\tau \in \Lambda$.

We introduce the following notation for some terms to be used below:

I$\equiv \lambda x.x$, T$\equiv \lambda xy.x$, F$\equiv \lambda xy.y$, $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$, if t_1 then t_2 else $t_3 \equiv t_1t_2t_3$, Zero$\equiv \lambda x.x$T, $\langle\perp\rangle \equiv \Omega, \langle 0\rangle \equiv$ I, $\langle n+1\rangle \equiv \lambda x.xF\langle n\rangle$, where $x, y \in V$, $t_1, t_2, t_3 \in \Lambda$, $n \in N$.

It is easy to see that: the term Ω does not have a head normal form, if T then t_2 else $t_3 =_\beta t_2$, if F then t_2 else $t_3 =_\beta t_3$, Zero$\langle 0\rangle =_\beta$T, Zero$\langle n+1\rangle =_\beta$F, Zero$\langle\perp\rangle$ does not have a head normal form, the term $\langle n\rangle$ is closed normal form, and if $n_1 \neq n_2$, then $\langle n_1\rangle$ and $\langle n_2\rangle$ are not congruent, $\langle n\rangle$ TII$=_\beta$I, where $n, n_1, n_2 \in N$.

Let us introduce the notion of λ-definability for the arithmetical functions with indeterminate values of arguments. A function $\varphi : M^k \to M,\ k \geq 1$, is said to be λ-definable, if there exists a term $\Phi \in \Lambda$, such that for all $m_1, \dots, m_k \in M$ we have:

$\Phi\langle m_1\rangle \dots \langle m_k\rangle =_\beta \langle \varphi(m_1, \dots, m_k)\rangle$, if $\varphi(m_1, \dots, m_k) \neq \perp$ and

$\Phi\langle m_1\rangle \dots \langle m_k\rangle$ does not have a head normal form, if $\varphi(m_1, \dots, m_k) = \perp$.

In this case Φ is said to be a term, which λ-defines the function φ .

Theorem 3. Every λ-definable arithmetical function with indeterminate values of arguments is monotonic.

Proof. Let $\varphi : M^k \to M,\ k \geq 1$, and $\Phi \in \Lambda$ λ-defines the function φ. Let $(m_1, \dots, m_k) \subseteq (\mu_1, \dots, \mu_k)$, where $m_i, \mu_i \in M,\ i = 1, \dots, k$, let us prove that $\varphi(m_1, \dots, m_k) \subseteq \varphi(\mu_1, \dots, \mu_k)$. For $\varphi(m_1, \dots, m_k) = \perp$, it is clear that

$\varphi(m_1,\dots,m_k) \subseteq \varphi(\mu_1,\dots,\mu_k)$. Let $\varphi(m_1,\dots,m_k) \neq \perp$, then $\Phi\langle m_1\rangle\dots\langle m_k\rangle =_\beta \langle\varphi(m_1,\dots,m_k)\rangle$ and according to point 1 of the Corollary of the CR-theorem $\Phi\langle m_1\rangle...\langle m_k\rangle \to\!\!\to_\beta \langle\varphi(m_1,...,m_k)\rangle$. It is easy to see that, if $m_i \subseteq \mu_i$ and $m_i \neq \mu_i$, then $m_i = \perp, i = 1,\dots,k$. Since $\langle\varphi(m_1,\dots,m_k)\rangle \in NF$, then $\Phi\langle\mu_1\rangle\dots\langle\mu_k\rangle \to\!\!\to_\beta \langle\varphi(m_1,\dots,m_k)\rangle$, and $\Phi\langle\mu_1\rangle\dots\langle\mu_k\rangle =_\beta \langle\varphi(m_1,\dots,m_k)\rangle$, which means that $\Phi\langle\mu_1\rangle\dots\langle\mu_k\rangle$ has normal form. Therefore, $\varphi(\mu_1,...,\mu_k) \neq \perp, \Phi\langle\mu_1\rangle...\langle\mu_k\rangle =_\beta \langle\varphi(\mu_1,...,\mu_k)\rangle, \langle\varphi(\mu_1,...,\mu_k)\rangle \in NF$. According to the point 2 of the Corollary of the CR-theorem, $\langle\varphi(m_1,\dots,m_k)\rangle \equiv \langle\varphi(\mu_1,...,\mu_k)\rangle$, i.e. $\varphi(m_1,...,m_k) = \varphi(\mu_1,...,\mu_k)$ and $\varphi(m_1,...,m_k) \subseteq \varphi(\mu_1,...,\mu_k)$. □

Theorem 4. Every λ-definable arithmetical function with indeterminate values of arguments is computable.

Proof. Let $\varphi : M^k \to M$, $k \geq 1$, and $\Phi \in \Lambda$ λ-defines the function φ. Let us describe an algorithm, which computes the function φ for $m_1,\dots,m_k \in M$. At first, the term $\Phi\langle m_1\rangle\dots\langle m_k\rangle$ is constructed. If $\varphi(m_1,\dots,m_k) \neq \perp$, then $\Phi\langle m_1\rangle\dots\langle m_k\rangle$ has normal form $\langle\varphi(m_1,\dots,m_k)\rangle$ and $\Phi\langle m_1\rangle\dots\langle m_k\rangle \to\!\!\to_\beta \langle\varphi(m_1,\dots,m_k)\rangle$, then we get $\varphi(m_1,\dots,m_k)$. If $\varphi(m_1,\dots,m_k) = \perp$, then the term $\Phi\langle m_1\rangle\dots\langle m_k\rangle$ does not have normal form and the left β-reduction for the term $\Phi\langle m_1\rangle\dots\langle m_k\rangle$ will be endless, which corresponds to the endless functioning of the algorithm. □

Theorem 5. Every computable, naturally extended arithmetical function with indeterminate values of arguments is λ-definable.

Proof. Let $\varphi : M^k \to M$, $k \geq 1$, be a computable, naturally extended arithmetical function with indeterminate values of arguments. We will show that the function φ is λ-definable. Two cases are considered:

Case 1. $\text{Arg}(\varphi) = \emptyset$.

In this case φ is the completely undefined function and the term $\Phi \equiv \lambda x_1\dots x_k.\Omega x_1\dots x_k$ λ-defines the function φ.

Case 2. $\text{Arg}(\varphi) \neq \emptyset$.

Let $A = \text{Arg}(\varphi)$. Consider the function $\psi : A \to N$, where $\psi(n_1,\dots,n_k) = \varphi(n_1,\dots,n_k)$, if $(n_1,\dots,n_k) \in A$. It is obvious that ψ is a computable, arithmetical function in the classical sense. Therefore, ψ is λ-definable in the classical sense (Kleene's theorem, [5]), i.e. there exists a term Ψ such that $\Psi\langle n_1\rangle\dots\langle n_k\rangle =_\beta \langle\psi(n_1\dots n_k)\rangle$, when $(n_1,\dots,n_k) \in A$, and the term $\Psi\langle n_1\rangle\dots\langle n_k\rangle$ does not have a head normal form otherwise. Let $\Phi \equiv \lambda x_1\dots x_k.(x_1\text{TII})\dots(x_k\text{TII})(\Psi x_1\dots x_k)$. It is easy to see that the term Φ λ-defines the function φ. □

Theorem 6. There exist strong computable, monotonic, not naturally extended arithmetical functions with indeterminate values of arguments, which are not λ-definable.

Proof. Consider function $\& : M^2 \to M$, where for all $m_1, m_2 \in M$ we have

$$\&(m_1,m_2) = \begin{cases} 0, & \text{if} \quad m_1 = 0 \text{ or } m_2 = 0, \\ 1, & \text{if} \quad m_1,\ m_2 \neq \perp \text{ and } m_1, m_2 \geq 1, \\ \perp, & \text{otherwise.} \end{cases}$$

It is easy to see that the function $\&$ is a strong computable, monotonic, not naturally extended arithmetical function with indeterminate values of arguments. Let us show that the function & is not λ-definable. Assume, that the function & is λ-definable and the term Φ λ-defines the function &. Consider the term Φxy, where x,y are different variables, which are not free in Φ. Since $\&(0,\perp)=0$, we have $\Phi \mathrm{I}\Omega =_\beta \mathrm{I}$ and by the left β-reduction of the term Φxy we can not get a term t, in which y is the leftmost free entry of a variable, which is on the left of all β-redexes of the term t, since in this case the term $\Phi \mathrm{I}\Omega$ will not have a normal form. On the other hand, since $\&(\perp,0)=0$, we have $\Phi\Omega\mathrm{I}{=_\beta}\mathrm{I}$ and by the left β-reduction of the term Φxy, we can not get a term t, in which x is the leftmost free entry of a variable, which is on the left of all β-redexes of the term t, since in this case the term $\Phi\Omega\mathrm{I}$ will not have a normal form. Thus, by the left reduction of the term Φxy we can get the term I. Therefore, by the left reduction of the term $\Phi\Omega\Omega$ we can get the term I too, and $\Phi\Omega\Omega =_\beta \mathrm{I}$. This is a contradiction, since $\&(\perp,\perp)=\perp$ and the term $\Phi\Omega\Omega$ does not have a normal form. Therefore, the function & is not λ-definable. $\square$

Theorem 7. There exist strong computable, monotonic, not naturally extended arithmetical functions with indeterminate values of arguments, which are λ-definable.

Proof. Consider function *and* $: M^2 \to M$, where for all $m_1, m_2 \in M$ we have

$$and(m_1,m_2)=\begin{cases} 0, & \text{if} \quad m_1=0 \ \text{ or } m_1\neq\perp, m_1\geq 1, m_2=0,\\ 1, & \text{if} \quad m_1,\ m_2\neq\perp \text{ and } \ m_1,m_2\geq 1,\\ \perp, & \text{otherwise.}\end{cases}$$

It is easy to see that the function *and* is a strong computable, monotonic, not naturally extended arithmetical function with indeterminate values of arguments, and the term Φ λ-defines this function, $\Phi \equiv \lambda xy.$ if Zero x then $\langle 0\rangle$ else (if Zero y then $\langle 0\rangle$ else $\langle 1\rangle$). $\square$

The Theorem 8 follows from the Theorems 5, 6, 7.

Theorem 8. The class of computable, naturally extended arithmetical functions with indeterminate values of arguments is a proper subclass of the class of λ-definable arithmetical functions with indeterminate values of arguments, which is a proper subclass of the class of computable, monotonic arithmetical functions with indeterminate values of arguments.

The δ-Redex Problem for the Strong Computable, Monotonic Arithmetical Functions with Indeterminate Values of Arguments. Let $\varphi : M^k \to M,\ k\geq 1$, be a strong computable, monotonic arithmetical function with indeterminate values of arguments. An expression $\varphi(\upsilon_1,\dots,\upsilon_k)$, where υ_i is from M, or is a variable, $i=1,\dots,k$, is called a δ-redex, if the value of the expression $\varphi(\upsilon_1,\dots,\upsilon_k)$ is the same for all values of variables.

The δ-redex problem for the function φ is formulated as follow: is there an algorithm, which for any expression $\varphi(\upsilon_1,\dots,\upsilon_k)$, where υ_i is either from M, or is a variable, $i=1,\dots,k$, determines wether this expression is a δ-redex or not.

Theorem 9. There exist strong computable, naturally extended arithmetical functions with indeterminate values of arguments, for which the δ-redex problem is unsolvable.

Proof. Let $T_0, T_1, \ldots, T_n, \ldots$ be an effective numeration of Turing machines (see [4]), $n \geq 0$. Let define the function $f : M^2 \to M$.

$$f(x,y) = \begin{cases} \perp, & \text{if } x = \perp \text{ or } y = \perp, \\ \perp, & \text{if } x \neq \perp,\ y \neq \perp \text{ and Turing machine } T_x \\ & \quad \text{does not halt on 0 after } \leq y \text{ steps,} \\ 1, & \text{if } x \neq \perp,\ y \neq \perp, \text{ Turing machine } T_x \\ & \quad \text{halts on 0 after } \leq y \text{ steps.} \end{cases}$$

It is easy to see that f is a strong computable, naturally extended arithmetical function with indeterminate values of arguments. Now we will prove that the δ-redex problem is unsolvable for the function f. Let $n \in N$. If the Turing machine T_n does not hold on 0, then $f(n,y) = \perp$ for all values of the variable y, i.e. $f(n,y)$ is δ-redex. If the Turing machine T_n holds on 0, then $f(n,r) = 1$ for some $r \in N$, and $f(n,\perp) = \perp$, which means that $f(n,y)$ is not a δ-redex. Thus, the assumption of the solvability of the δ-redex problem for function f will imply the solvability of the halting problem for Turing machines. □

Corollary 1. (Theorem 9). There exist strong computable, monotonic arithmetical functions with indeterminate values of arguments, for which the δ-redex problem is unsolvable.

Corollary 2. (Theorem 9). There exist strong computable, λ-definable arithmetical functions with indeterminate values of arguments, for which the δ-redex problem is unsolvable.

Received 20.10.2014

REFERENCES

1. **Manna Z.** Mathematical Theory of Computation. McGraw-Hill Book Company, 1974.
2. **Nigiyan S.A.** Arithmetical Functions with Indeterminate Values of Arguments. Computability and λ-Definability. // National Academy of Sciences of Armenia. Reports, 2014, v. 114, № 1, p. 7–13 (in Russian).
3. **Malcev A.I.** Algorithms and Recursive Functions. M.: Nauka, 1986 (in Russian).
4. **Rogers H.** Theory of Recursive Functions and Effective Computability. MIT McGraw-Hill Book Company, 1967.
5. **Barendregt H.** The Lambda Calculus. Its Syntax and Semantics. Amsterdam: North-Holland Publishing Company, 1981.