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NON-CLASSICAL PROBLEM OF BEND OF AN ORTHOTROPIC ANNULAR PLATE OF VARIABLE THICKNESS WITH AN ELASTICALLY CLAMPED SUPPORT

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A model of elastically clamped support for an inner edge of an axisymmetric annular round bending plate is proposed. Values of parameters of the support as well as relationship between them is determined. With the collocation method the problem of bending of a cylindrically orthotropic annual plate of a variable thickness under a uniformly distributed load is solved taking into account the transverse shear. It is assumed that the inner edge of the plate is elastically fastened and the outer one is hingedly supported. Based on the analysis of calculated values of immeasurable quantities qualitative conclusions are drawn.

MSC2010: 74K20.

***Keywords*:** elastically clamped support, cylindrical orthotropy, variable thickness, transverse shear.

Introduction. Extensive use of elastically clamped supports in building structures has led to the practical need for their studies. In the literature the following works are dedicated to the theoretical and practical aspects of the model of elastically clamped supports of thin-walled structural elements and their applications [1–15].

1. Problem Statement. Let us consider an elastically clamped support (Fig. 1) in the right cylindrical coordinate system r, θ, z.

The part of the ring plate's inner edge is inserted in an elastically deformable array. The length of the inserted part by the radius d is sufficiently small compared to the radius R_1 of the inner edge of the plate. Because of this, the inserted part practically will move vertically and rotate around its center of mass as one whole like an absolutely solid element. Therefore, the value of the derivative $\dfrac{dw}{dr}$ of the deflection within the inserted part is assumed to be constant. For simplicity, we will

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assume that the plate is deformed by axisymmetric bending. Then, on its internal contour $r = R_1$ only shear force N_r and bending moment M_r arise. The vanishing of the static moment of the inserted part's differential element with respect to its center of mass is given by

$$\int_0^a (R_1 - a + x)\,x\,dx - \int_0^{d-a} (R_1 - a - x)\,x\,dx = 0, \qquad (1.1)$$

a denotes the distance of the inserted part's center of mass from the inside edge of the plate $r = R_1$. After some calculations from (1.1) we obtain

$$a = \frac{d\,(3R_1 - 2d)}{3\,(2R_1 - d)}. \qquad (1.2)$$

From (1.2) we see, that if $d \ll R_1$, then $a \approx \frac{d}{2}$. This occurs, for example, with elastically clamped support of the outer contour of the plate. Under the action of the moment aN_r of the shear force about the inserted part's center of mass, and the moment M_r of the plate's edge, the plate's median plane will rotate through a certain angle in the support section $r = R_1$. The tangent $\frac{dw}{dr}$ of this angle depends on the rotating moments. It is easy to notice that the positive sign of $\frac{dw}{dr}$ in the adopted right coordinate system corresponds to the negative moments. With this in mind, and assuming that the value of $\frac{dw}{dr}$ is directly proportional to the sum of the moments, we can write:

$$\left.\frac{dw}{dr}\right|_{r=R_1} = D\,(aN_r - M_r)\,. \qquad (1.3)$$

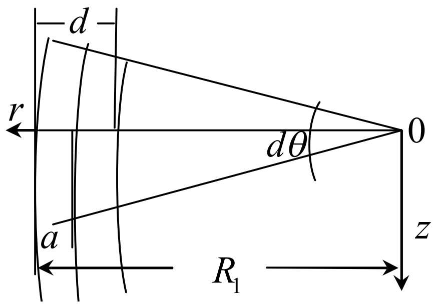

Fig. 1.

The positive constant D is the reciprocal of the stiffness of the elastically clamped support on rotation. It has dimension N^{-1} in the SI system. The inner contour's $r = R_1$ deflection consists of two parts. One occurs by the rotation of the inserted part around its center of mass, and the other one is because of the shear force on the contour. Assuming that the second part is directly proportional to the shear force, it is possible to write

$$w\Big|_{r=R_1} = \left(a\frac{dw}{dr} + BN_r\right). \qquad (1.4)$$

The constant B is the reciprocal of the stiffness of the elastically clamped support on vertical offset. It has dimension m^2N^{-1} in the SI system. Thus, (1.3) and (1.4) are the conditions of the elastically clamped support of the plate's internal edge when it is axisymmetrically bent. They are essentially the same with the conditions of elastically clamped beam supports, experiencing lateral bending deformation [1].

2. Obtaining the Parameters Connection. Let us define the parameters B and D of the elastically clamped support and the relationship between them. We use Fuss–Winkler hypothesis, according to which the deformed elastic array acts on the inserted part of the plate with stresses that are directly proportional to the arised displacements. As a result, a vertical force acts on the inserted part of the plate, which balances the marginal shear force $N_rR_1d\theta$. The moment of the forces (exerted by the elastic solid) about the center of mass of the inserted part is to balance the sum of the moments $(aN_r - M_r)R_1d\theta$. These conditions are written as

$$2k_1\int_0^d (R_1-d+x)dx + h_0k_2(R_1-d) = \frac{R_1}{B}, \tag{2.1}$$

$$2k_1\left[\int_0^a (R_1-a+x)x^2dx + \int_0^{d-a}(R_1-a-x)x^2dx\right] + h_0k_2(R_1-d)(d-a)^2 = \frac{R_1}{D}, \tag{2.2}$$

here h_0 is the constant thickness of the plate's inserted part; k_1 and k_2 are coefficients of proportionality of normal and tangential contact stresses respectively. They have dimension $N\cdot m^{-3}$ in the SI system. For the parameters of the elastically clamped support of the plate's inner edge, from (2.1) and (2.2) we obtain:

$$B = \frac{R_1}{k_1d(2R_1-d)+h_0k_2(R_1-d)}, \tag{2.3}$$

$$D = \frac{6R_1}{k_1d\left[d^2(4R_1+8a-3d)-6a(ad+2R_1d-2aR_1)\right]+6k_2h_0(R_1-d)(d-a)^2}. \tag{2.4}$$

In the case where the inserted part's end does not contact the elastically deformable array, the members of the expressions (2.3) and (2.4) with factor k_2 will be omitted. Then the parameters D and B of the elastically clamped support are in the following relationship:

$$D = \frac{6(2R_1-d)B}{d^2(4R_1+8a-3d)-6a(ad+2R_1d-2aR_1)}. \tag{2.5}$$

This exact relationship is used below in the solution of the particular task of axisymmetric bending of the plate.

3. Problem Statement. Consider a cylindrically orthotropic circular ring plate with inner and outer radii R_1 and R_2 respectively. The inner edge of the small plate of length d and with constant thickness h_0 is inserted into the elastically deformable array, forming an elastically clamped support. The outer edge of the plate is hinged.

The plate bears a uniformly distributed shear load of intensity q. The plate thickness varies linearly with the following formula:

$$h = h_0 + h_1 (r - R_1), \quad R_2 \geq r \geq R_1, \tag{3.1}$$

here h_0 and h_1 are given constants.

4. Basic Notation and Obtaining the System of Resolving Equations.

We use the notation:

$$\begin{gathered} r = R_2\rho, \ \ R_1 = kR_2, \ \ s = \frac{h_0}{R_2}, \ \ h = h_0 H, \ \ H = 1 + \gamma(\rho - k), \\ \gamma = \frac{h_1}{s}, \ \ w = h_0\bar{w}, \ \ q = B_r\bar{q}, \ \ B_\theta = mB_r, \\ \varphi_1 = B_r\bar{\varphi}_1, \ \ a_r B_r = \chi, \ \ d = h_0\bar{d}, \ \ a = h_0\bar{a}, \ \ N_r = B_r h_0 \bar{N}_r, \\ M_r = B_r h_0^2 \bar{M}_r, \ \ M_\theta = B_r h_0^2 \bar{M}_\theta, \ \ B = \frac{\bar{B}}{B_r}, \ \ D = \frac{\bar{D}}{B_r h_0^2}, \end{gathered} \tag{4.1}$$

here B_r, B_θ, a_r are well-known mechanical properties of the material [2]; χ is determines the effect of transverse shear; φ_1 is characterizes the change of the transverse shear of the plate. With the notation (4.1) the expressions (1.2) and (2.5) take the form:

$$\bar{a} = \frac{\bar{d}\left(3k - 2s\bar{d}\right)}{3\left(2k - s\bar{d}\right)}, \tag{4.2}$$

$$\bar{D} = \frac{6\left(2k - s\bar{d}\right)\bar{B}}{\bar{d}^2\left(4k + 8s\bar{a} - 3s\bar{d}\right) - 6\bar{a}\left(\bar{a}s\bar{d} + 2k\bar{d} - 2k\bar{a}\right)}. \tag{4.3}$$

In view of (4.1) for the expression of the shear forces and bending moments, taking into account the effect of the shear [3], we obtain:

$$\bar{N}_r = \frac{H}{12}\left[8\bar{\varphi}_1 - \gamma s^2 H\left(s\frac{d^2\bar{w}}{d\rho^2} + \frac{\nu_\theta}{\rho}s\frac{d\bar{w}}{d\rho} - \chi\frac{d\bar{\varphi}_1}{d\rho} - \frac{\nu_\theta}{\rho}\chi\bar{\varphi}_1\right)\right], \tag{4.4}$$

$$\bar{M}_r = -\frac{sH^3}{12\rho}\left(s\rho\frac{d^2\bar{w}}{d\rho^2} + \nu_\theta s\frac{d\bar{w}}{d\rho} - \chi\rho\frac{d\bar{\varphi}_1}{d\rho} - \nu_\theta\chi\bar{\varphi}_1\right), \tag{4.5}$$

$$\bar{M}_\theta = -\frac{msH^3}{12\rho}\left(s\frac{d\bar{w}}{d\rho} + \rho s\nu_r\frac{d^2\bar{w}}{d\rho^2} - \chi\nu_r\rho\frac{d\bar{\varphi}_1}{d\rho} - \chi\bar{\varphi}_1\right). \tag{4.6}$$

Taking into account (4.1), the equilibrium differential equations of axisymmetric bending of the plate [3] will take the following dimensionless forms:

$$\begin{gathered} \gamma s^4\nu_\theta\rho H^2\frac{d^2\bar{w}}{d\rho^2} + ms^4\gamma H^2\frac{d\bar{w}}{d\rho} - sH\rho\left(8\rho + \chi\nu_\theta\gamma H s^2\right)\frac{d\bar{\varphi}_1}{d\rho} - \\ -s\left(8\rho H + 16\gamma\rho^2 + \chi m\gamma s^2 H^2\right)\bar{\varphi}_1 = 12\rho^2\bar{q}, \end{gathered} \tag{4.7}$$

$$\begin{gathered} s^3\rho^2H^2\frac{d^3\bar{w}}{d\rho^3} + s^3\rho H(H + 2\gamma\rho)\frac{d^2\bar{w}}{d\rho^2} + s^3H(2\nu_\theta\gamma\rho - mH)\frac{d\bar{w}}{d\rho} - \\ -\chi s^2H^2\rho^2\frac{d^2\bar{\varphi}_1}{d\rho^2} - \chi s^2H\rho(H + 2\gamma\rho)\frac{d\bar{\varphi}_1}{d\rho} + \\ +\left(8\rho^2 + \chi m s^2H^2 - 2\chi\nu_\theta H\gamma\rho s^2\right)\bar{\varphi}_1 = 0. \end{gathered} \tag{4.8}$$

The system of Eqs. (4.7), (4.8) are fourth-order. Taking into account (4.1), the boundary conditions are:

$$\bar{w} = \bar{a}s\frac{d\bar{w}}{d\rho} + \bar{B}\bar{N}_r, \quad s\frac{d\bar{w}}{d\rho} = \bar{D}\left(\bar{a}\bar{N}_r - \bar{M}_r\right), \quad \rho = k, \tag{4.9}$$

$$\bar{w} = 0, \quad \rho s\frac{d^2\bar{w}}{d\rho^2} + \nu_\theta s\frac{d\bar{w}}{d\rho} - \rho\chi\frac{d\bar{\varphi}_1}{d\rho} - \nu_\theta\chi\bar{\varphi}_1 = 0, \quad \rho = 1. \tag{4.10}$$

The problem will be solved by the collocation method [16]. Let

$$\bar{w} = a_0 + \sum_{i=1}^{n} a_i\rho^i, \quad \bar{\varphi}_1 = b_0 + \sum_{i=1}^{n} b_i\rho^i, \tag{4.11}$$

here $a_0,\ a_i,\ b_0,\ b_i$ are unknown constants. We will use the Eqs. (4.7), (4.8) and boundary conditions (4.9) and (4.10) to determine their values. The interval $k < \rho < 1$ will be splitted into n parts. The number of splitting points is equal to $(n-1)$. By writing the system of Eqs. (4.7), (4.8) for these points, we obtain $2(n-1)$ equations. Adding the four boundary conditions (4.9) and (4.10), we obtain the system of algebraic equations for the $2(n+1)$ unknown constants. By increasing n to such a value where the calculation process almost converges, we obtain the solution of the problem. Dimensionless values of the calculation quantities of the plate are defined with the formulas (4.1)–(4.6).

5. Considered Example. Since the calculation quantities of the plate are directly proportional to the value of the intensity of the load, then for simplicity we assume that $\bar{q} = 1$. In each case, we will multiply the solution by the real value $\bar{q}$ and will obtain the actual values of the calculation quantities.

Let: $\bar{d} = 1,\ s = 0.05,\ \gamma = 1,\ \nu_\theta = 0.2,\ \nu_r = 0.4,\ k = 0.2,\ m = 0.5,$ $\chi = 0,\ 5,\ \bar{q} = 1\,(\bar{a} = 0.4762)\,,\ \bar{D} = 12.0822\,\bar{B}.$

Table 1

		ρ				
		0.2	0.4	0.6	0.8	1
$\bar{w}$	$n = 8$	3.6807	1575.1	2347.8	1564.2	0
	$n = 10$	3.6921	1581.2	2353.0	1573.2	0
$\bar{N}_r$	$n = 8$	25.164	8.3190	1.7961	–2.1694	–9.2933
	$n = 10$	23.033	7.8993	1.9267	–2.0538	–7.4308
$\bar{M}_r$	$n = 8$	–47.617	9.5831	21.713	16.548	0
	$n = 10$	–49.201	9.4128	21.793	16.930	0
$\bar{M}_\theta$	$n = 8$	–7.6126	-1.0992	4.6391	6.4051	4.4257
	$n = 10$	–8.0776	–1.1080	4.6384	6.4936	4.4798

One can see from Tab. 1, that $\bar{B} = 0.01,\ \gamma = 1,\ \chi = 5$ convergence at $n = 10$. This is the case with other values of the problem parameters too.

For clarity, in the Figs. 2–5 are shown the graphs of the variation of dimensionless calculation quantities for some values of the parameters of the problem.

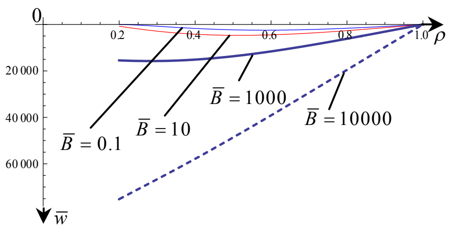

Fig. 2.

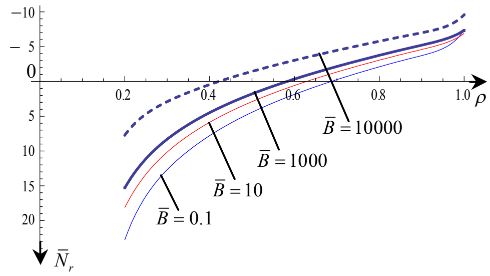

Fig. 3.

Table 2

				ρ				
				0.2	0.4	0.6	0.8	1
$\bar{B}=0.001$	$\gamma=0$	$\chi=0$	$\bar{w}$	4.2464	2709.9	4893.8	3706.3	0
			$\bar{N}_r$	24.878	8.1245	2.0734	–1.9434	–7.4468
			$\bar{M}_r$	–57.636	4.7252	18.744	14.696	0
			$\bar{M}_\theta$	–11.608	-3.1379	3.2908	4.6081	1.9856
		$\chi=5$	$\bar{w}$	4.4999	2785.6	5064.9	3875.9	0
			$\bar{N}_r$	23.347	8.3615	2.2383	–1.8208	–5.7149
			$\bar{M}_r$	–63.038	3.4599	18.527	15.032	0
			$\bar{M}_\theta$	–11.015	–3.2234	3.2520	4.6881	2.0338
	$\gamma=1$	$\chi=0$	$\bar{w}$	3.1266	1351.1	1957.2	1257.9	0
			$\bar{N}_r$	26.300	6.8154	1.1799	–2.6093	–16.123
			$\bar{M}_r$	–37.247	10.568	19.612	12.863	0
			$\bar{M}_\theta$	–7.5071	–0.7141	4.2497	5.2998	3.6235
		$\chi=5$	$\bar{w}$	3.6921	1581.2	2353.0	1573.2	0
			$\bar{N}_r$	23.033	7.8993	1.9267	–2.0538	–7.4308
			$\bar{M}_r$	–49.201	9.4128	21.793	16.930	0
			$\bar{M}_\theta$	–8.0776	–1.1080	4.6384	6.4936	4.4798
$\bar{B}=0.002$	$\gamma=0$	$\chi=0$	$\bar{w}$	8.4660	2730.8	4913.5	3717.5	0
			$\bar{N}_r$	24.854	8.1169	2.0683	–1.9472	–7.4426
			$\bar{M}_r$	–57.419	4.8068	18.782	14.712	0
			$\bar{M}_\theta$	–11.644	–3.1276	3.3031	4.6175	1.9911
		$\chi=5$	$\bar{w}$	8.9708	2807.7	5085.5	3887.4	0
			$\bar{N}_r$	23.327	8.353	2.2326	–1.8251	–5.7162
			$\bar{M}_r$	–62.798	3.5478	18.567	15.047	0
			$\bar{M}_\theta$	–11.054	–3.2122	3.2649	4.6977	2.0394
	$\gamma=1$	$\chi=0$	$\bar{w}$	6.2265	1364.1	1968.3	1263.9	0
			$\bar{N}_r$	26.265	6.8084	1.1754	–2.6127	–16.094
			$\bar{M}_r$	–37.039	10.655	19.657	12.887	0
			$\bar{M}_\theta$	–7.5225	–0.6998	4.2677	5.3189	3.6407
		$\chi=5$	$\bar{w}$	7.3492	1596.1	2365.2	1579.5	0
			$\bar{N}_r$	23.004	7.8874	1.9188	–2.0597	–7.4277
			$\bar{M}_r$	–48.918	9.5170	21.837	16.944	0
			$\bar{M}_\theta$	–8.0931	–1.0900	4.6579	6.5116	4.4968

				ρ				
				0.2	0.4	0.6	0.8	1
$\bar B = 0.01$	$\gamma = 0$	$\varkappa = 0$	$\bar w$	41.288	2892.7	5066.2	3804.4	0
			$\bar N_r$	24.666	8.0573	2.0289	–1.9769	–7.4098
			$\bar M_r$	–55.729	5.4410	19.078	14.837	0
			$\bar M_\theta$	–11.927	–3.0473	3.3987	4.6908	2.0339
		$\varkappa = 5$	$\bar w$	43.723	2978.8	5245.4	3977.2	0
			$\bar N_r$	23.173	8.2857	2.1879	–1.8587	–5.7260
			$\bar M_r$	–60.932	4.2293	18.876	15.167	0
			$\bar M_\theta$	–11.354	–3.1258	3.3651	4.7717	2.0827
	$\gamma = 1$	$\varkappa = 0$	$\bar w$	30.109	1464.0	2054.1	1310.5	0
			$\bar N_r$	25.997	6.7547	1.1402	–2.6392	–15.869
			$\bar M_r$	–35.435	11.323	20.003	13.077	0
			$\bar M_\theta$	–7.6406	–0.5901	4.4057	5.4658	3.7734
		$\varkappa = 5$	$\bar w$	35.419	1710.2	2458.2	1627.6	0
			$\bar N_r$	22.783	7.7962	1.8582	–2.1052	–7.4036
			$\bar M_r$	–46.753	10.315	22.176	17.053	0
			$\bar M_\theta$	–8.2119	–0.9521	4.8072	6.6494	4.6275
$\bar B = 10$	$\gamma = 0$	$\varkappa = 0$	$\bar w$	1118.0	7662.9	9539.5	6343.4	0
			$\bar N_r$	19.244	6.3319	0.8855	–2.8355	–6.4665
			$\bar M_r$	–6.9231	23.768	27.639	18.448	0
			$\bar M_\theta$	–20.011	–0.6875	6.1852	6.8276	3.2837
		$\varkappa = 5$	$\bar w$	1159.6	7934.4	9848.8	6557.9	0
			$\bar N_r$	18.786	6.3742	0.9156	–2.8132	–6.0059
			$\bar M_r$	–7.9449	23.603	27.664	18.572	0
			$\bar M_\theta$	–19.796	–0.6295	6.2379	6.8951	3.3281
	$\gamma = 1$	$\varkappa = 0$	$\bar w$	711.64	3813.9	4047.1	2389.3	0
			$\bar N_r$	19.942	5.5362	0.3441	–3.2394	–10.821
			$\bar M_r$	0.5937	26.319	27.779	17.339	0
			$\bar M_\theta$	–10.189	1.9559	7.5846	8.8479	6.8434
		$\varkappa = 5$	$\bar w$	759.31	4197.3	4470.1	2665.6	0
			$\bar N_r$	18.138	5.8773	0.5826	–3.0626	–6.9042
			$\bar M_r$	–1.4076	27.034	29.276	19.322	0
			$\bar M_\theta$	–10.604	2.0108	8.0059	9.6122	7.4475

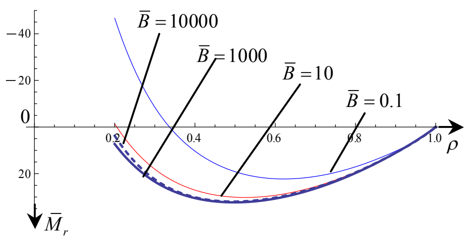

Fig. 4.

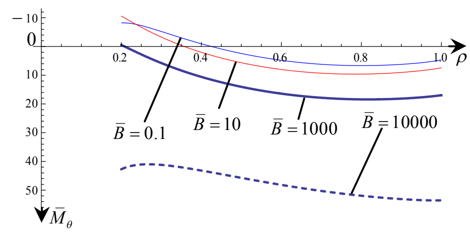

Fig. 5.

Conclusion. The data in Tab. 2 and graphs 2–5 lead to the following conclusions:

1. With the increase of the parameter $\bar{B}$, and hence also $\bar{D}$, the stiffness of the elastically clamped support reduces, resulting in an increased deflection of the plate.

2. Weakening elastically clamped support did not significantly affect the qualitative behavior of the calculation quantities of the plate along radial coordinate ρ.

3. A substantial weakening of elastically clamped support leads to substantial increase of the deflection of clamped edge of the plate. The shear force N_r and the bending moment M_r are decreased. The fixed edge of the plate tends to the free edge.

4. As expected, when $\chi > 0$, that is, taking into account the effect of transverse shear, the deflections of the plate increases.

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