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Mathematics

DESCRIPTION OF ORDER 3 HYPERGROUPS OVER GROUP ARISING FROM DIHEDRAL GROUP D_9

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In the present paper we describe, up to isomorphism, all order 3 unitary hypergroups over group, arising from the dihedreal group D_9.

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Introduction. Completely reduced hypergroups of order 3 over a group can be obtained only either from the symmetric group S_3 or from noncommutative groups of order 18 [1]. The unitary order 3 hypergroups over group, arising from S_3, are studied in [2]. It is well-known that there exist only three non-isomorphic non-commutative groups of order 18, namely, $S_3 \times C_3$, the dihedral group D_9 and the semidirect product of $C_3 \times C_3$ with C_2 (see, e.g., [3]). In [1] it is proved that any hypergroup of order 3 over group, arising from the group $S_3 \times C_3$, is reducible. In this paper we describe, up to isomorphism, all unitary hypergroups of order 3 over group, arising from the dihedral group D_9.

Hypergroups Over Group. The notion of hypergroup over a group was introduced in [4]. The general form of the definition of hypergroup over a group is as follows.

Let H be an arbitrary (multiplicative) group with the neutral element (identity) ε. A (right) hypergroup over H is a set M together with the system of structural mappings Ω=$(\Phi, \Psi, \Lambda, \Xi)$, satisfying the following conditions:

(B1) The structural mapping Ξ is a binary operation on M :

$$\Xi : M \times M \longrightarrow M, \quad (ab) \longrightarrow [a,b],$$

determining on M a structure of right loop with a left neutral element o .

(B2) The structural mapping Φ is a right action of the group H on M:

$$\Phi : M \times H \longrightarrow M, \quad a\alpha \longrightarrow a^{\alpha}.$$

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(B3) The structural mapping Λ is a scalar product on M with values in H:

$$\Lambda : M \times M \longrightarrow H, \quad ab \longrightarrow (a,b).$$

We denote $\theta = (o,o)^{-1}$.

(B4) The structural mapping Ψ is defined by

$$\Psi : M \times H \longrightarrow H, \quad a\alpha \longrightarrow {}^{a}\alpha.$$

Moreover, the following list of axioms must hold:

$$(\mathbf{A}1) \quad {}^{a}(\alpha\beta) = {}^{a}\alpha \cdot {}^{a^{\alpha}}\beta,$$

$$(\mathbf{A}2) \quad [a,b]^{\alpha} = [a^{({}^{b}\alpha)}, b^{\alpha}],$$

$$(\mathbf{A}3) \quad {}^{[a,b]}\alpha = (a,b)^{-1} \cdot {}^{a}({}^{b}\alpha) \cdot (a^{({}^{b}\alpha)}, b^{\alpha}),$$

$$(\mathbf{A}4) \quad [[a,b],c] = [a^{(b,c)}, [b,c]],$$

$$(\mathbf{A}5) \quad (a,b)([a,b],c) = {}^{a}(b,c)(a^{(b,c)}, [b,c]),$$

$$(\mathbf{A}6) \quad {}^{o}\alpha = \theta^{-1} \cdot \alpha \cdot \theta,$$

We denote a (right) hypergroup M over the group H by ${}_{H}M$.

A morphism ϕ: ${}_{H}M \longrightarrow {}_{H'}M'$ is defined as a pair (ϕ_0, ϕ_1), consisting a homomorphism of groups ϕ_0: $H \longrightarrow H'$ and a map of sets ϕ_1: $M \longrightarrow M'$, preserving all structural mappings. The hypergroups over groups and their morphisms form a category [5]. Therefore, one can use the notions and terminology of general category theory (isomorphism, epimorphism etc).

Theorem 1. Let G be an arbitrary group, H be its subgroup and M be a complementary set to H (i.e. a section σ of canonical surjection ψ: $G \longrightarrow H\backslash G$). Then the system of mappings Ω=(Φ, Ψ, Λ, Ξ) defined by conditions

$$a \cdot \alpha = {}^{a}\alpha \cdot a^{\alpha}, \quad a \cdot b = (a,b)[a,b], \quad a,b \in M, \alpha \in H,$$

gives a structure of hypergroup on M. Conversely, any hypergroup ${}_{H}M$ can be obtained in this way [6].

A hypergroup ${}_{H}M$ is called unitary, if $\sigma(H \cdot e) = e$, where e is the neutral element of G.

A representation $\underline{\Phi}$: $H \longrightarrow S_M$ is canonically associated with any hypergroup ${}_{H}M$. The hypergroup ${}_{H}M$ is called completely reduced, if the kernel of this representation is trivial [7].

The Dihedral Group D_9. The dihedral group of order 18 is given by two generators a and b and relations

$$a^9 = b^2 = e, \quad ab = ba^{-1}.$$

It is a semidirect product of the cyclic subgroup $\langle a \rangle$ of order 9 with the cyclic subgroup $\langle b \rangle$ of order 2.

Proposition 1. The dihedral group D_9 has exactly 3 subgroups of order 6:

$$H_1=\langle\ a^3,\ b\ \rangle=\{\ e,\ a^3,\ a^6,\ b,\ a^3b,\ a^6b\ \},$$

$$H_2=\langle\ a^3,\ ab\ \rangle=\{\ e,\ a^3,\ a^6,\ ab,\ a^4b,\ a^7b\ \},$$

$$H_3=\langle\ a^3,\ a^2b\ \rangle=\{\ e,\ a^3,\ a^6,\ a^2b,\ a^5b,\ a^8b\ \}.$$

These subgroups are isomorphic to the symmetric group S_3.

Let J be the set of first 9 non-negative integers and $I = \{1,2,4,5,7,8\}$.

Theorem 2. The group of automorphisms $A = Aut\ D_9$ has order 54 and consists of elements $\varphi_{ij},\ i \in I$ and $j \in J$, defined by

$$\varphi_{ij}\ (a^k b^l)=a^{(ik+jl)(\mathrm{mod}\ 9)}b^l.$$

The law of binary operation in A is:

$$\varphi_{i,j}\cdot\varphi_{k,l}=\varphi_{m,n}, \qquad m \equiv ik\ (\mathrm{mod}\ 9)\ \text{ and }\ n \equiv jk+l\ (\mathrm{mod}\ 9).$$

This group is the exact product of the cyclic subgroup $\langle\alpha\rangle$ of order 6, generated by $\alpha=\varphi_{2,0}$, and the cyclic subgroup $\langle\beta\rangle$ of order 9, generated by $\beta = \varphi_{1,1}$.

The Equivalence Classes of Sections of Canonical Surjections, Associated with Subgroups of Order 6 of Dihedral Group D_9. Let G be a group, H and H' be its subgroups,

$$\psi : G \longrightarrow H\setminus G \text{ and } \psi' : G \longrightarrow H'\setminus G$$

be the canonical surjections, M and M' be the sections of ψ and ψ', respectively. We say that the sections M and M' are equivalent, if there exists an automorphism α of G such that $\alpha(H)=H'$ and $\alpha(M)=M'$. Evidently, this relation is an equivalence relation. The dihedral group D_9 has exactly three subgroups H_1, H_2, H_3 of order 6 and they are conjugate each to other. Consequently, every section of canonical surjection, associated with an order 6 subgroup of D_9 is equivalent to a section of canonical surjection, associated with, for example, H_1. Further we denote

$$H = H_1 = \langle a^3,b\rangle = \{e,a^3,a^6,b,a^3b,a^6b\},$$

and let $\psi=\psi_1$ be the associated canonical surjection.

One of the three cosets of the quotient-set $H\backslash D_9$ is H, and the two others are

$$\{a,a^4,a^7,a^2b,a^5b,a^8b\}, \quad \{a^2,a^5,a^8,ab,a^4b,a^7b\}.$$

Three first elements of these cosets have order 9, and three last elements have order 2. The surjection ψ has 36 unitary sections:

(i) 9 sections contain two elements of order 2;

(ii) 9 sections contain two elements of order 9;

(iii) 18 sections contain one elements of order 2 and one element of order 9.

Evidently, two sections M and $M^{'}$ can be equivalent, only if there exists a bijection between M and $M^{'}$ such that the corresponding elements have the same order.

Proposition 2. The set B of all automorphisms φ of D_9 satisfying $\varphi(H) = H$ is a subgroup of order 18. It consists of elements

$$B = \{\varphi_{i,j};\ i \in I \ \text{ and } \ j = 0,3,6\}.$$

Proposition 3. Any two unitary sections of ψ, containing two elements of order 2, are equivalent.

Proposition 4. There exist two classes of equivalent unitary sections of ψ, containing two elements of order 9:

1. $\{ e, a, a^2 \}, \{ e, a^2, a^4 \}, \{ e, a^4, a^8 \}, \{ e, a^5, a \}, \{ e, a^7, a^5 \}, \{ e, a^8, a^7 \}$.
2. $\{ e, a, a^8 \}, \{ e, a^4, a^5 \}, \{ e, a^7, a^2 \}$.

Proposition 5. Any two unitary sections of ψ, containing one element of order 2, and one element of order 9, are equivalent.

Order 3 Hypergroups, Arising from Dihedral Group D_9.

Theorem 3. Let G be an arbitrary group, H and $H^{'}$ be subgroups of G with equivalent the complementary subsets M to H and $M^{'}$ to $H^{'}$. Then the hypergroups ${}_HM$ and ${}_{H'}M^{'}$ are isomorphic.

We have, up to equivalence, exactly four sections for canonical surjection, associated to all subgroups H_i of D_9. Therefore, we can consider only the subgroup $H = H_1$ and, for example, the sections M_1={ e, a^2b, ab }, M_2={ e, a, a^2 }, M_3={ e, a^4, a^5 } and M_4={ e, a, ab }.

Proposition 6. Let Ω_i=(Φ_i, Ψ_i, Λ_i, Ξ_i) be the system of structural mappings of hypergroup ${}_HM_i$ (i=1,2,3,4). These structural mappings are given by the following tables:

1) $i = 1$

Φ_1	e	a^3	a^6	b	a^3b	a^6b
e	e	e	e	e	e	e
a^2b	a^2b	a^2b	a^2b	ab	ab	ab
ab	ab	ab	ab	a^2b	a^2b	a^2b

Ψ_1	e	a^3	a^6	b	a^3b	a^6b
e	e	a^3	a^6	b	a^3b	a^6b
a^2b	e	a^6	a^3	a^3b	b	a^6b
ab	e	a^6	a^3	a^3b	b	a^6b

Λ_1	e	a^2b	ab
e	e	e	e
a^2b	e	e	a^3b
ab	e	b	e

Ξ_1	e	a^2b	ab
e	e	a^2b	ab
a^2b	a^2b	e	a^2b
ab	ab	ab	e.

Thus, the hypergroup ${}_HM_1$ is reduced to a hypergroup over the group C_2. The corresponding right loop (M_1, Ξ_1) is not associative.

2) $i = 2$

Φ_2	e	a^3	a^6	b	a^3b	a^6b
e	e	e	e	e	e	e
a	a	a	a	a^2	a^2	a^2
a^2	a^2	a^2	a^2	a	a	a

Ψ_2	e	a^3	a^6	b	a^3b	a^6b
e	e	a^3	a^6	b	a^3b	a^6b
a	e	a^3	a^6	a^3b	a^6b	b
a^2	e	a^3	a^6	a^3b	a^6b	b

Λ_2	e	a	a^2
e	e	e	e
a	e	e	a^3
a^2	e	a^3	a^3

Ξ_2	e	a	a^2
e	e	a	a^2
a	a	a^2	e
a^2	a^2	e	a

This hypergroup $_HM_2$ is reduced to a hypergroup over the group C_2. The corresponding right loop (M_2, Ξ_2) is a group.

3) $i = 3$

Φ_3	e	a^3	a^6	b	a^3b	a^6b
e	e	e	e	e	e	e
a^4	a^4	a^4	a^4	a^5	a^5	a^5
a^5	a^5	a^5	a^5	a^4	a^4	a^4

Ψ_3	e	a^3	a^6	b	a^3b	a^6b
e	e	a^3	a^6	b	a^3b	a^6b
a^4	e	a^3	a^6	b	a^3b	a^6b
a^5	e	a^3	a^6	b	a^3b	a^6b

Λ_3	e	a^4	a^5
e	e	e	e
a^4	e	a^3	e
a^5	e	e	a^6

Ξ_3	e	a^4	a^5
e	e	a^4	a^5
a^4	a^4	a^5	e
a^5	a^5	e	a^4

This hypergroup $_HM_3$ is isomorphic to $_HM_2$.

4) $i = 4$

Φ_4	e	a^3	a^6	b	a^3b	a^6b
e	e	e	e	e	e	e
a	a	a	a	ab	ab	ab
ab	ab	ab	ab	a	a	a

Ψ_4	e	a^3	a^6	b	a^3b	a^6b
e	e	a^3	a^6	b	a^3b	a^6b
a	e	a^3	a^6	e	a^3	a^6
ab	e	a^6	a^3	e	a^6	a^3

Λ_4	e	a	ab
e	e	e	e
a	e	a^3b	a^3b
ab	e	b	e

Ξ_4	e	a	ab
e	e	a	ab
a	a	ab	a
ab	ab	e	e

This hypergroup is also reduced to hypergroup over C_2. The corresponding right loop (M_4,Ξ_4) is not associative and is not isomorphic to the right loop (M_1, Ξ_1).

Theorem 4. All hypergroups of order 3, arising from dihedral group D_9, are defined over the symmetric group S_3 up to isomorphism. There exist only three such unitary hypergroups. They are reduced to three (non-isomorphic) unitary hypergroups of order 3, arising from S_3.

Note that, the converse assertion to Theorem 3 is not true, according to Propositions 4 and 6. The sections M_2 and M_3 are not equivalent, but the hypergroups ${}_HM_2$ and ${}_HM_3$ are isomorphic.

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