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M a t h e m a t i c s

ON ALGEBRAS OF BOUNDED FUNCTIONS ON COMPLETELY REGULAR SPACES

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Some aspects on strict topology in algebras of continuous and bounded functions on completely regular spaces are discussed in this work. Also some properties of their dual spaces are investigated.

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Introduction. Let Ω be a topological space and $B(\Omega)$ be the algebra of all bounded complex-valued functions on Ω, $C(\Omega)$ be the algebra of all complex-valued continuous functions on Ω, and let $C_\infty(\Omega)$ be the subalgebra of $B(\Omega)$ containing all continuous functions.

Note that within isometric isomorphism we have $B_b^*(\Omega) = \mathcal{M}(\Omega)$, where $\mathcal{M}(\Omega)$ is the space of all bounded regular measures, defined on the σ-algebra of the subsets $\Sigma = 2^\Omega$ of the set Ω. Here $B_b(\Omega)$ is the Banach algebra of all bounded complex-valued functions on Ω in the sup-norms (see [1]).

Denote by $B_0(\Omega)$ the ideal in algebra $B(\Omega)$ consisting of the functions vanishing at "infinity" (i.e. for any $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ such that $|f(x)| < \varepsilon$ for $x \in \Omega \setminus K$). Then $C_0(\Omega) = B_0(\Omega) \cap C_\infty(\Omega)$ is an ideal in $C_\infty(\Omega)$.

Let $B_{00}(\Omega)$ be the ideal in the algebra $B(\Omega)$ consisting of those functions, supports of which are compact subsets of Ω. Clearly, $B_{00}(\Omega) \subset B_0(\Omega)$ and $C_{00}(\Omega) = B_{00}(\Omega) \cap C_\infty(\Omega)$ (see [1–4]).

Note that the ideal $B_0(\Omega)$ contains:

1) all characteristic functions χ_K, where $K \subset \Omega$ is compact;

2) all functions g of the following form $g(x) = \sum\limits_{n=1}^{\infty} \alpha_n \chi_{K_n}(x)$, where K_n are mutually disjoint compacts in Ω, $\alpha_n \in \mathbb{C}$ and $|\alpha_n| \to 0$ as $n \to \infty$;

3) functions of the form $g(x) = \sum\limits_{n=1}^{\infty} \alpha_n \chi_{\{x_n\}}(x)$, where $(x_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of points from Ω and $\alpha_n \in \mathbb{C}$, $|\alpha_n| \to 0$ as $n \to \infty$.

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Let Ω be a Hausdorff space and $\mathcal{B}(\Omega)$ be the σ-algebra of Borel subsets of Ω (i.e. σ-algebra generated by closed subsets of Ω). In what follows, $\mathcal{M}(\Omega)$ is the space of all finite complex-valued regular measures on $(\Omega, \mathcal{B}(\Omega))$.

Proposition 1. If Ω is a topological space and f is a complex-valued function on Ω such that $fg \in B_0(\Omega)$ whenever $g \in B_0(\Omega)$, then $f \in B(\Omega)$.

Proof. Suppose $f \notin B(\Omega)$. Choose a sequence $(x_n) \subset \Omega$ such that $|f(x_n)| > n$. Consider the function $g \in B_0(\Omega)$, which is defined by the formula

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cdot \frac{\overline{f(x_n)}}{|f(x_n)|^2} \chi_{\{x_n\}}(x).$$

Since

$$(gf)(x_n) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cdot \frac{\overline{f(x_n)}\, f(x_n)}{|f(x_n)|^2} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty,$$

then $gf \notin B(\Omega)$, which contradicts the condition in the Proposition 1. Therefore, $f \in B(\Omega)$. $\square$

Note that in Property 1 the algebra $B(\Omega)$ can be replaced by algebra $L^\infty(\Omega)$, where ideal $B_0(\Omega)$ is replaced by ideal $L_0^\infty(\Omega)$.

Recall (see [5]) that Hausdorf's space Ω is called:

a) *locally compact space*, if each point in Ω has a precompact neighborhood;

b) *completely regular*, if for every set $E \subset \Omega$ and for each point $x_0 \in \Omega$, which is not an adherent point for E, there exists a continuous function on Ω, which is zero on E and is different from zero in the point x_0.

Corollary 1. Let Ω be a complete regular space and the function $f \in C(\Omega)$ be such that for every $g \in C_0(\Omega)$ the function $fg \in C_\infty(\Omega)$. Then $f \in C_\infty(\Omega)$ (see [5, 6]).

Proof. Suppose $f \notin C_\infty(\Omega)$. Then there exists a sequence $(x_n) \in \Omega$ such that $|f(x_n)| > n$. Let the function $g \in C_0(\Omega)$ be such that $g(x_n) = \frac{1}{\sqrt{n}} \cdot \frac{\overline{f(x_n)}}{|f(x_n)|}$. Since $(fg)(x_n) = \frac{1}{\sqrt{n}} |f(x_n)| > \sqrt{n}$. This contradiction shows that $f \in C_\infty(\Omega)$. $\square$

Proposition 2. Let the measure μ is positive and σ-finite. If f is a complex-valued function such that $gf \in L^1(\Omega, \mu)$ for all $g \in L^1(\Omega, \mu)$, then $f \in L^\infty(\Omega, \mu)$.

Proof. Suppose that $f \notin L^\infty(\Omega, \mu)$. Then there exists a sequence of nonintersecting measurable sets $\{E_n\}$ with positive finite measures such that $|f(x)| \geqslant n^{1+\varepsilon}$ for each $x \in E_n$, where $\varepsilon > 0$. We define the functions g by:

$$g(x) = \begin{cases} \dfrac{1}{|f(x)|\mu(E_n)}, & \text{if } \ x \in E_n \ (n \in \mathbb{N}), \\ 0, & \text{if } \ x \in \Omega \setminus \bigcup\limits_n E_n. \end{cases}$$

Then

$$\int\limits_\Omega |g|\, d\mu = \sum_n \int\limits_\Omega \frac{d\mu}{|f|\mu(E_n)} \leqslant \sum_n \frac{1}{n^{1+\varepsilon}} \int\limits_{E_n} \frac{d\mu}{\mu(E_n)} \leqslant \sum_n \frac{1}{n^{1+\varepsilon}} < \infty,$$

and so

$$\int\limits_{\Omega} |gf|\, d\mu = \sum_n \int\limits_{E_n} \frac{d\mu}{\mu(E_n)} = \infty.$$

Then we get $gf \notin L^1(\Omega,\mu)$, hence $f \in L^\infty(\Omega,\mu)$.

Note that every locally compact space is a complete regular space. It is well-known (see [6, 7]) that every complete regular space Ω admits a compact extension. Therefore, in the sequal it is natural to suppose, that Ω is completely regular space.

Going back to the general case, we let Ω be any set. Then every function $g \in B(\Omega)$ defines a seminorm P_g on the algebra $B(\Omega)$ by the formula

$$P_g(f) = \|fg\|_\infty = \sup_{x\in\Omega} |f(x)g(x)|.$$

If $\Phi \subset B_0(\Omega)$ is an arbitrary family of functions, then Φ-topology on the $B(\Omega)$ algebra is the topology defined by the system of seminorms $\{P_g\}_{g\in\Phi}$. $\big($The base of open neighborhoods of zero for the Φ-topology is the system $\{U_g\}_{g\in\Phi}$, where $U_g = \{f \in B(\Omega) : P_g(f) < 1\}\big)$.

Let Ω be a completely regular space and $\Phi = B_0(\Omega)$. Then the topology on the $B(\Omega)$ algebra defined by the family of seminorms $\{P_g\}_{g\in B_0(\Omega)}$ is called β-topology, and the $B(\Omega)$ algebra with this topology is denoted by $B_\beta(\Omega)$. As it was mentioned above, if Ω is a completely regular compact space, then the topology in $C_\infty(\Omega)$ defined by the family of seminorms $\{P_g\}_{g\in C_0(\Omega)}$ is a β-topology, and corresponding topological algebra is denoted by $C_\beta(\Omega)$ (see [1, 2]). Clearly we have $C_\beta(\Omega) \subset B_\beta(\Omega)$. Note, that if Ω is a completely regular space, then the topology in $C_\infty(\Omega)$ should be defined using the seminorms generated by the $B_0(\Omega)$ ideal.

If we define a topology using sup-norm in the algebra $C_\infty(\Omega)$, then we will have a Banach algebra $C_b(\Omega)$ with the topology of uniform convergence on Ω.

Actually, the β-topology on $C_\infty(\Omega)$, generated by the family of seminorms $\{P_g\}_{g\in B_0(\Omega)}$, is the topology generated by a family of functions of type 2), i.e. by the seminorms of the form

$$P_g(f) = \left\| \sum_{n=1}^{\infty} \alpha_n \chi_{K_n} f \right\|_\infty = \sup_{n\geqslant 1} \left\{ |\alpha_n|\, \|f\|_{K_n} \right\}.$$

Recall (see [5, 6]), that the closed subalgebra $\mathcal{A}$ of the algebra $C_\beta(\Omega)$ in the β-topology is called β-uniform, if it contains constant functions and separates the points of Ω $\big($i.e. for any $x_1, x_2 \in \Omega$, $x_1 \neq x_2$ there exists a function $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)\big)$. In this case β-uniform algebra is denoted by $\mathcal{A}_\beta$. Since the uniform topology is stronger than β-topology, the β-uniform algebra is a uniform closed subalgebra of the algebra $C_b(\Omega)$. The algebra $\mathcal{A}$, equipped with the uniform topology, is denoted by $\mathcal{A}_\infty$ as it was mentioned above, and the space of its maximal ideals is denoted by $M_{\mathcal{A}_\infty}$ (see [7, 8]).

Recall that the β-uniform algebra $\mathcal{A}_\beta$ is called *regular*, if for every closed set $F \subset M_{\mathcal{A}_\infty}$ and a point outside of that set $x \in M_{\mathcal{A}_\infty}$ there exists a function $g \in \mathcal{A}_\beta$ such that $g(x) = 0$ for every $x \in F$ and $g(x_0) \neq 0$.

Notice, that if Ω is the completely regular space, then the algebra $C_\beta(\Omega)$ is complete topological algebra (see [5,6]).

Theorem 1. Let Ω be a complete regular space and $\mathcal{A}_\beta$ be a symmetric regular β-uniform subalgebra of the $C_\beta(\Omega)$ algebra. Then $\mathcal{A}_\beta^* = \mathcal{M}(\Omega)$.

Proof. Let φ be a β-continuous linear functional on $\mathcal{A}_\beta$ algebra. Then it will be a continuous functional on the Banach algebra $\mathcal{A}_\infty = C(M_{\mathcal{A}_\infty})$, since the algebra $\mathcal{A}_\beta$ is regular and symmetric. By Riesz theorem there exists a finite regular Borel measure μ on $M_{\mathcal{A}_\infty}$, which is the representing measure for φ, i.e.

$$\varphi(f) = \int\limits_{M_{\mathcal{A}_\infty}} \hat{f}\, d\mu\,,$$

where $\hat{f}$ is the Gelfand transform of f. Represent the measure μ as a sum $\mu = \mu_{\Omega} + \mu_{F_\infty}$, where $M_{\mathcal{A}_\infty} = \Omega \cup F_\infty$, μ_Ω and μ_{F_∞} is the restriction of the measure μ on Ω and F_∞ respectively. Proof that $\mu_{F_\infty} = 0$.

Let $\{e_i\}_{i\in I}$ be the bounded approximate identity in $C_0(\Omega)$. $\big($Note that since the algebra $\mathcal{A}$ is regular and symmetric, the approximate identity exists also in $\mathcal{A}_0 = \mathcal{A}_\beta \cap C_0(\Omega)\big)$. As it was shown in [5,6], the net of functions $\{f_i\}_{i\in I}$, where $f_i = 1 - e_i$, converges to the zero function in Ω with β-topology on $\mathcal{A}_\beta$. Hence, the net of functionals $(f_i \circ \varphi)_{i\in I}$, where $(f_i \circ \varphi)(f) = \varphi(f_i f)$ converges to the zero functional.

Thus,

$$0 = \lim_I (f_i \circ \varphi)(f) = \lim_I \left(\int\limits_{\Omega} \widehat{f_i f}\, d\mu + \int\limits_{F_\infty} \widehat{f_i f}\, d\mu \right) = \int\limits_{F_\infty} \hat{f}\, d\mu$$

for any $f \in \mathcal{A}_\infty$. Since $\mathcal{A}_\beta$ is symmetric and regular, we have $\mu_{F_\infty} = 0$. Thus, for any β-continuous linear functional on $\mathcal{A}_\beta$ there is a measure from $\mathcal{M}(\Omega)$ corresponding to that functional. The Proof of the converse statement is obvious. $\square$

Corollary 2. Let $\mathcal{A}_\beta$ be a symmetric regular β-uniform subalgebra of the $C_\beta(\Omega)$ algebra. Then there exists a linear multiplicative functional $\varphi : \mathcal{A}_\beta \to \mathbb{C}$, which is discontinuous in β-topology.

Proof. Note, that if Ω is compact, then the space of multiplicative functionals coincides wiith Ω, since $M_{\mathcal{A}_\infty} = \Omega$, i.e. every multiplicative functional φ is a Dirac functional δ_x, $x \in \Omega$. Note that φ has the unique representative measure on Ω, which coincides with the Dirac atomic measure centred at some point x_0. On the other hand, if Ω is a completely regular space, then $\mathcal{A}_\infty$ is isomorph isometric to $C(M_{\mathcal{A}_\infty})$.

Let $x_0 \in M_{\mathcal{A}_\infty} \setminus \Omega$. Then the multiplicative functional $\varphi : C(M_{\mathcal{A}_\infty}) \to \mathbb{C}$ satisfies $\varphi(f) = \hat{f}(x_0)$ and is also a multiplicative functional on the $\mathcal{A}_\beta$ algebra. Since the functional φ has the a unique representative measure concentrated at the point x_0, by Theorem 1, the functional φ is not continuous.

Note that from Proposition 1 the commutative Banach algebras $B_b(\Omega)$ and $L^\infty(\Omega,\mu)$ can be transformed into β-uniform algebras if the topologies are introduced by the family of seminorms $\{P_g\}_{g\in B_0(\Omega)}$ and $\{P_g\}_{g\in L_0^\infty(\Omega,\mu)}$ respectively, where $P_g(f) = \sup\limits_{\Omega} |fg|$, $f \in B(\Omega)$ and $P_g(f) = \operatorname{ess}\sup\limits_{\Omega} |fg|$, $f \in L^\infty(\Omega, \mu)$. Then for topological algebras $B_\beta(\Omega)$ and $L^\infty_\beta(\Omega, \mu)$ we have the following results:

Theorem 2. Let Ω be a complete regular space. Then topological algebras the $B_\beta(\Omega)$ and $L^\infty_\beta(\Omega, \mu)$ are complete in the corresponding topologies.

Theorem 3. Let Ω be a complete regular space and $\mathcal{A}_\beta$ be a symmetric regular β-uniform subalgebra of the algebra $B_\beta(\Omega)$. Then $\mathcal{A}^*_\beta = \mathcal{M}(\Omega)$.

Note that in the Theorem 3 the algebra $B_\beta(\Omega)$ can be replaced by the topological algebra $L^\infty_\beta(\Omega, \mu)$.

Note that for topological algebras $B_\beta(\Omega)$ and $L^\infty_\beta(\Omega, \mu)$, the notion of β-amenable algebras can be introduced and the analogues of the M.B. Sheinberg's Theorem can be obtained.

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