PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2017, **51**(1), p. 42–45

Mechanics

ON THE CONDITION OF PLANAR LOCALIZED VIBRATIONS APPEARANCE IN THE VICINITY OF THE FREE EDGE OF A THIN RECTANGULAR PLATE

M. V. BELUBEKYAN *

Institute of Mechanics of NAS of Republic of Armenia

On the basis of the equations of generalized plane stress state the problems of planar vibrations of a rectangular thin plate are investigated. It is established possibility of the appearance of vibrations with amplitude decreasing exponentially, when moving from free edge to the opposite fixed edge. The conditions of the appearance of such localized vibrations depending on the size of the plate and the methods of fixation of other three sides are obtained.

MSC2010: 74J15.

***Keywords*:** plane stress state, Navier conditions, sliding contact, clumped edge.

Introduction. Several works are dedicated to the problems of wave propagation along the free edge of the semi-infinite thin plate (see, for example [1, 2]). The investigations of edge waves are conducted on the base of the equations of generalized plane stress state. At three-dimensional statement, the edge waves are considered in [3]. In the case of plates with limited dimensions, perhaps more suitable is the expression "vibration, localized in the vicinity of the free edge".

In the paper it is shown how it is possible to obtain the dispersion equation of the edge waves with limiting transition from the solution of the problem for rectangular plates.

1. Thin elastic plate occupied (in the Cartesian coordinate system) an area $0 \le x \le a,\ 0 \le y \le b,\ -h \le z \le h$. The equations of plate are following [4, 5]:

$$\Delta u + \theta_1 \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{1}{c_t^2}\cdot\frac{\partial^2 u}{\partial t^2}; \quad \Delta v + \theta_1 \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{1}{c_t^2}\cdot\frac{\partial^2 v}{\partial t^2}, \tag{1}$$

here $u,\ v$ are planar displacement of median plane $(z = 0)$ of plate, Δ is two-dimensional Laplace operator

$$c_t^2 = G/\rho, \quad \theta_1 = (1+\nu)/(1-\nu), \tag{2}$$

where G is shear modulus, ν is Poisson ratio, ρ is the plate material density, c_t is pure shear wave velocity.

The expressions for the forces in the plate are following:

$$T_1 = C\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right),\ T_2 = C\left(\frac{\partial v}{\partial x} + \nu\frac{\partial u}{\partial y}\right),\ S = \frac{1-\nu}{2}C\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{3}$$

where $C = 2Eh/(1-\nu^2)$ is a rigidity of plate on tension (compression), E is Young's modulus.

* E-mail: mbelubekyan@yahoo.com

It is known that with the help of the Lame transformations [5, 6]

$$u = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}, \; v = \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x}. \tag{4}$$

The system of Eqs. (1) is reduced to autonomous equations:

$$\frac{\partial^2 \varphi}{\partial t^2} = c_e^2 \Delta\varphi, \;\; \frac{\partial^2 \psi}{\partial t^2} = c_t^2 \Delta\psi, \;\; c_e^2 = \frac{E}{(1-v^2)\rho}. \tag{5}$$

It is assumed that at the edges of plate $y = 0,\, b$ there are the Navier-type conditions, $T_2 = 0,\, u = 0$. It is easy to show that these conditions for harmonic vibrations of plates (exactly what is considered here) are equivalent to conditions $\partial\psi/\partial y = 0,\; \varphi = 0$ for $y = 0,\, b$ [7].

The solutions of Eqs. (5) satisfying the mentioned above boundary conditions are presented in the form:

$$\varphi = e^{i\omega t} \sum_{n=1}^{\infty} \varphi_n(x) \sin\lambda_n y, \;\; \psi = e^{i\omega t} \sum_{n=1}^{\infty} \psi_n(x) \cos\lambda_n y, \;\; \lambda_n = n\pi/b. \tag{6}$$

Substitution (6) into Eq. (5) leads to a system of consistent equations relatively functions $\varphi_n(x),\; \psi_n(x)$ common solutions of which are as follows:

$$\varphi_n = A_n \sinh\lambda_n p_1 x + B_n \cosh\lambda_n p_1 x, \quad \psi_n = C_n \sinh\lambda_n p_2 x + D_n \cosh\lambda_n p_2 x, \tag{7}$$

where $A_n,\; B_n,\; C_n,\; D_n$ are arbitrary constants, $p_1 = \sqrt{1-\theta\eta},\; p_2 = \sqrt{1-\eta},\; \eta = \dfrac{\omega^2}{\lambda_n^2 c_t^2}$, $\theta = \dfrac{1-v}{2}$.

It is further assumed that the edge $x = 0$ is free, i.e $T_1 = 0,\; S = 0$ under $x = 0$. These conditions, taking into account the transformations (4) and submission (6), reduced to the form

$$\varphi_n'' - v\lambda_n^2 \varphi_n - \lambda_n(1-v)\psi_n' = 0, \; 2\lambda_n\varphi_n' - \lambda_n^2\psi_n - \psi_n'' = 0 \;\text{ under }\; x = 0. \tag{8}$$

2. On the remaining edge of the plate $x = 0$ would be considered various options for the boundary conditions. Let on the fourth edge of the plate, as well as on the edges $y = 0,\, b$ the Navier conditions occur, i.e. $T_1 = 0, v = 0$ under $x = a$. These conditions, taking into account the conversion of the Lame (4), correspond to the conditions $\varphi = 0,\; \dfrac{\partial\psi}{\partial x} = 0$.

Demanding, that the general solutions of (7) satisfy the boundary conditions above, it is easy to obtain:

$$\varphi_n = A_n \sinh\lambda_n p_1(a-x), \;\; \psi_n = D_n \cosh\lambda_n p_2(a-x), \tag{9}$$

where A_n and D_n are new arbitrary constants.

Substitution (9) into the boundary conditions of the free edge (8) leads to a homogeneous system of algebraic equations relatively to the arbitrary constants

$$\begin{gathered}(p_1^2 - v)A_n \sinh\lambda_n p_1 a + (1-v)p_2 D_n \sinh\lambda_n p_2 a = 0,\\ 2p_1 A_n \cosh\lambda_n p_1 a + (1+p_2^2)D_n \cosh\lambda_n p_2 a = 0.\end{gathered} \tag{10}$$

The equality to zero of the determinant of the system (10) leads to the equation that defines the dimensionless parameter of vibrations frequency η

$$(2-\eta)^2 \tanh\lambda_n p_1 a - 4p_1 p_2 \tanh\lambda_n p_2 a = 0. \tag{11}$$

This equation in the limit $a \to \infty$ ($\tanh\lambda_n p_1 a \to 1,\; \tanh\lambda_n p_2 a \to 1$) leads to the Rayleigh equation (see [6]). Eq. (11) as Rayleigh equation, always (regardless of the attitude of the plate sides) has a solution satisfying the condition

$$0 < \eta < 1. \tag{12}$$

Accordingly, for a rectangular plate with three sides having the boundary conditions of the Navier-type with one free side, there are vibrations, localized in the vicinity of the free edge.

The second variant of the boundary conditions are the conditions as anti-Navier type (or sliding contact)

$$S=0,\;\; u=0 \text{ under } x=a. \tag{13}$$

The condition (13) regarding the function $\varphi,\ \psi$ are following:

$$\partial\varphi/\partial x=0,\;\; \psi=0 \text{ under } x=a. \tag{14}$$

From general solution (7) the following solution satisfying the conditions (14) is obtained:

$$\varphi_n=B_n\cosh\lambda_n p_1(a-x),\;\; \psi_n=C_n\sinh\lambda_n p_2(a-x). \tag{15}$$

After satisfaction of the boundary conditions of the free edge (8) is obtained

$$\begin{gathered}\left(p_1^2-\nu\right)B_n\cosh\lambda_n p_1 a+(1-\nu)p_2 C_n\cosh\lambda_n p_2 a=0,\\ 2p_1 B_n\sinh\lambda_n p_1 a+\left(1+p_2^2\right)C_n\sinh\lambda_n p_2 a=0.\end{gathered} \tag{16}$$

The condition of equality to zero of the determinant of (16) system reduces to the form

$$(2-\eta)^2\tanh\lambda_n p_2 a-4p_1p_2\tanh\lambda_n p_1 a=0. \tag{17}$$

In this case, for $\lim\limits_{a\to\infty}\lambda_n a\to\infty$ the equation of Rayleigh is obtained. However, here a solution satisfying the condition (12) does not exist for all $\lambda_n a$.

If Eq. (17) divides into p_2 and $\eta\to 1 (p_2\to 0)$, the limited equation would be obtained

$$\lambda_n a-\sqrt{1-\theta}\tanh\left(\lambda_n a\sqrt{1-\theta}\right)=0. \tag{18}$$

Designating the root of the Eq. (18) $(\lambda_n a)_*=\xi_*$ conditions for the occurrence of solutions, which satisfying the condition (12) (the appearance of localized vibrations), we obtain $\lambda_n a>\xi_*$.

In particular, at approximation $\tanh\left(\lambda_n a\sqrt{1-\theta}\right)\approx 1$ obtains

$$\lambda_n a>2\sqrt{2(1+\nu)}. \tag{19}$$

A similar approach for finding condition of solution localization used in [8].

3. Let the edge of plate $x=a$ is fixed. We have $u=0,\ v=0$ under $x=a$, or, taking into account the Eqs. (4), (6), $\varphi_n^1-\lambda_n\psi=0,\ \lambda_n\varphi_n-\psi_n^1=0$ under $x=a$.

Substitution (7) into the last boundary conditions, leads to the following system of algebraic equations for the arbitrary constants A_n,B_n,C_n,D_n :

$$\begin{gathered}p_1A_n\cosh\lambda_n p_1 a+p_1B_n\sinh\lambda_n p_1 a-C_n\sinh\lambda_n p_2 a-D_n\cosh\lambda_n p_2 a=0,\\ A_n\sinh\lambda_n p_1 a+B_n\cosh\lambda_n p_1 a-p_2C_n\cosh\lambda_n p_2 a-p_2D_n\sinh\lambda_n p_2 a=0.\end{gathered} \tag{20}$$

For the system (20) joining the equations that are obtained from the requirements of satisfying the boundary conditions of free edge $x=0$ (2)

$$(2-\eta)B_n-2p_2C_n=0,\quad 2p_1A_n-(2-\eta)D_n=0. \tag{21}$$

The condition of equality to zero of the equation system (20) and (21) determinant, brought to the equation

$$\begin{gathered}4(2-\eta)p_1p_2-\left[(2-\eta)^2+4\right]p_1p_2\cosh\lambda_n p_1 a\cosh\lambda_n p_2 a+\\ +\left[(2-\eta)^2+4p_1^2p_2^2\right]\sinh\lambda_n p_1 a\sinh\lambda_n p_2 a=0.\end{gathered} \tag{22}$$

For $a\to\infty$ at condition (12), after some transformations the Eq. (22) brought to

$$(1-p_1p_2)\left((2-\eta)^2-4p_1p_2\right)=0.$$

From (22) follows the Rayleigh equation, i.e. $p_1p_2\neq 1$.

By dividing the Eq. (22) on p_2 and letting η to unit $(p_2 \to 0)$, we obtained an equation that determines the parameter $\lambda_n a$ leading to the appearance of localized vibrations:

$$4\sqrt{1-\theta} - 5\sqrt{1-\theta}\cosh\left(\sqrt{1-\theta}\lambda_n a\right) + \lambda_n a \sinh\left(\sqrt{1-\theta}\lambda_n a\right) = 0. \qquad (23)$$

Particularly, in approximation $\lambda_n a \gg 1$ the condition of occurrence of localized waves in the following form is obtained $\lambda_n a > 5\sqrt{1-\theta}$. In compare with the case where at the edge $x = a$ there are the conditions of sliding contact (19), the parameter $\lambda_n a$ is increased 1.25 times.

Conclusion. The problems of vibration of thin plate are considered, when at two opposite edges the Navier conditions are given, the third part is free and on the fourth, free opposite side are assumed three options: Navier conditions, sliding contact, rigid fixation. It is established that in the case of the Navier conditions on the fourth side, localized vibrations in the vicinity of the free edge, exist always. For the other two cases the conditions of the appearance of localized vibrations depending on the plate side relations and Poisson's ratio are obtained.

Received 15.09.2016

REFERENCES

1. **Belubekyan M.V.** Surface Waves in Elastic Media. // The Problems of Solid Mechanics. Inst. of Mechanics NAS RA, 1997, p. 79–96 (in Russian).
2. **Wilde M.V., Kaplunov J.D., Kossovich L.Yu.** Boundary and Interface Resonance Phenomena in Elastic Bodies. M.: Fizmatlit, 2010, 280 p. (in Russian).
3. **Ardazishvili R.V.** Influence of the Method of Fixing of Front Surface on the Damping of the Antisymmetric Edge Waves of Higher Order in the Plates. XI Congress on Fundamental Problems of Theoretical and Applied Mechanics. Kazan, 2015 (in Russian).
4. **Ambartsumian S.A.** Theory of Anisotropic Plates. M.: Nauka, 1987, 360 p. (in Russian).
5. **Belubekyan M.V.** The Equations of the Theory of Plates, Taking Into Account the Transverse Shears. In: Problems of Mechanics of Thin Deformable Bodies. Yer.: Acadimia Press, 2002, p. 67–88 (in Russian).
6. **Nowacki W.** Theory of Elasticity. M.: Mir, 1975, 872 p. (in Russian).
7. **Belubekyan V.M., Belubekyan M.V.** The 3D Problem of the Elastic Wave Guide with Rectangular Cross Section. // Proceeding of NAS RA. Mechanics, 2015, v. 68, № 4, p. 3–8 (in Russian).
8. **Tovstik P.E.** The Stability of Thin Shells. Asymptotic Methods. M.: Nauka, 1995, 320 p. (in Russian).