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M a t h e m a t i c s

In Memory of Hayk Badalyan.
To the Centennial of Birth

USING GAUGING IN NONLINEAR PROBLEMS OF ECONOMICS, PHYSICS AND TECHNOLOGY

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The approach proposed in this paper uses ideas and instruments of gauging theory to handle nonlinear problems in economics, nonlinear dynamics and various technological issues. We see nonlinearity of gauging as hardly a technical issue arising while solving the variation problem, but a structural feature of an intrinsically nonlinear space shaped by its gauge-based connections. Under these assumptions, the solution of the variation problem is shown as invariant to transformations of the vector field which present the initial source of nonlinear disturbances.

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Introduction. Our equations in general are similar to those of parallel transport, which are well known in differential geometry. In the simplest cases, they are identical, i.e. Eq.(6). Note that they were obtained within the variational framework Eqs. (1)−(8), which is more generic, thus, we believe, more promising than parallel transport approach, which is typically used. Also, the use of variational technique focuses us on the role of connections, which can be useful in empirical fields from physics to economics and technology. The major difference is the emphasis placed on the shaping of the connections as a means to handle practical cases using appropriate gauging functions. This is the main feature of our approach. We see the connections, typically affine and expressed through the Christoffel symbols, as principal source and chief instrument defining the shape and the direction of any specific nonlinear

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effects. Closely related to geometry, our approach allows maximum flexibility for interdisciplinary applications. As shown in this paper, it can be used in such diverse areas as economics, physics and technology by bridging together the equations of economic dynamics and physics along with various technological approaches.

1. Gauges and How to Use Them in and Outside of Physics.

• *Gauges* are the basic measuring functions used generically for observation purposes.

• *The main idea:* a variable is presented as a parametric vector in the base space of e_i

$$Y = Y^i e_i. \tag{1}$$

• *The Gaussian gauge* is among popular examples used in the pattern recognition

$$e^{\sum \frac{(x_i^0 - x_i)^2}{2}}. \tag{2}$$

In the most generic sense, it represents a non-linear function of distance.

2. Gauges were First Developed in Physics:

• *As settings* for focusing equations to assure adequate group of transformations.

• *Aiming* to focus on the signal by filtering out the white noise. In pattern recognition this contributes to separability the ability to distinguish processed images as true positives while avoiding mixing them with false positives.

• *Compensation for nonlinearity*, assuring linearization and invariant measurement. Technically, invariance of vector in a gauge basis equals to finding the extremum for a functional

$$Y = Y^i e_i, \tag{3}$$

since

$$\delta Y = 0, \tag{4}$$

which is the necessary condition for equilibrium.

The equilibrium means a mutual compensation of all extant flows at a given point as the precondition for the very existence of an invariant system of measurement. The latter is necessary both for experiment reproducibility and the very ability to measure.

3. Repurposing Gauges for the Economics.

Conditions for Gauging. Let's assume that the vector of supply Y can be gauged through base demand e_i

• *The Equilibrium Hypothesis.* Eq. (4) is the mathematical representation of the general equilibrium, achievable by varying prices. This is the main issue of the classical economics and the bone of contention between warring econ. schools [1–4].

• *Conditions for Clearing* all the flows at each and every point of sale can be derived from Eq. (4) and rewritten as

$$\delta Y^i e_i = -\delta e_i Y^i,\ \frac{\delta Y^\alpha}{Y^\alpha} = -\frac{\Delta e_\alpha}{e_\alpha},\ \Delta e_\alpha = \left(\frac{(\delta e_i)^\alpha}{Y^\alpha} Y^i\right) e_\alpha,\ Y^i e_i = \max, \tag{5}$$

Δe_α is directional compensation, $(\delta e_i)^\alpha = \Gamma^\alpha_{ij} \delta x^j$ is components of δe_i along e_α.

This means that any increase in supply must be compensated by a corresponding increase in demand by varying prices at the point of sale [5–8].

From Theoretical Considerations to Real Equations. Assuming $x = x(t)$, $x = x^0,\ x^1, \dots, x^N,\ x^0 = t,\ 0 \le i < N$, the extremum for the functional Eq. (3) in its simplest form can be done through differentiation of $Y^k = Y^k(x),\ e_k = e_k(x)$, $0 < k \le K$. Differentiation brings us to $\dfrac{d(Y^k e_k)}{dt} = 0$ and we get

$$\left(\frac{\partial Y^k}{\partial x^j} e_k + \frac{\partial e_k}{\partial x^j} Y^k\right) \dot{x}^j = \left(\frac{\partial Y^k}{\partial x^j} e_k + \frac{\partial e_i}{\partial x^j} Y^i\right) \dot{x}^j = \left(\frac{\partial Y^k}{\partial x^j} + \Gamma^k_{ij} Y^i\right) e_k \dot{x}^j. \quad (6)$$

Assuming the independence of base vectors e_k and variables $\dot{x}^j$, we get the equations of parallel transport [9]

$$\left(\frac{\partial Y^k}{\partial x^j} + \Gamma^k_{ij} Y^i\right) = 0, \qquad \left(\dot{Y}^k + \Gamma^k_{ij} \dot{x}^j Y^i\right) = 0, \quad (7)$$

where, by definition,

$$\Gamma^k_{ij} e_k = \frac{\partial e_i}{\partial x^j}, \quad \Gamma^m_{ij} e_m e^k = \frac{\partial e_i}{\partial x^j} e^k, \quad \Gamma^k_{ij} = \frac{\partial e_i}{\partial x^j} e^k \quad (8)$$

are the connections or Christoffel's symbols [9].

Gauges–Popular Systems and Simple Examples. Exponents $e^{\pm\phi}$, where $\phi(x,t)$ is the cumulate, x is the growth factor, t is the time. Exponents $e_i = e^{\pm\phi_i}$, $\phi_i(x,t) = \int \nu_i(x,\dot{x},t)dt$. The base equations are produced by gauging $e_i = e^{-\phi_i}$, $\phi_i = \int \nu_i(x,\dot{x},t)dt$, considering $\dfrac{\partial e_i}{\partial x^j} = -\dfrac{\partial \phi_k}{\partial x^j} \delta^k_i e_k$. From (8) we get $\Gamma^\alpha_{ij} = -\dfrac{\partial \phi_\alpha}{\partial x^j} \delta^\alpha_i$ with non-zero $\Gamma^{\alpha_j}_\alpha = \dfrac{\partial \phi_\alpha}{\partial x^j}$. Substituting $\Gamma^{\alpha_j}_i$ in (7), we obtain simple base equations

$$\frac{\partial Y_\alpha}{\partial x^i} - \frac{\partial \phi_\alpha}{\partial x^i} Y^\alpha = 0, \quad \dot{Y}^\alpha - \nu_\alpha Y^\alpha = 0, \quad \nu_\alpha = (k_\alpha \dot{x} - \omega_\alpha). \quad (9)$$

Where, assuming $x = x^0, x^1, \dots, x^N, x^0 = t, k_\alpha = (k_{\alpha_1}, \dots, k_{\alpha_N}), 0 \le \alpha < K, 0 \le i < N$,

$$\frac{\partial \phi_\alpha}{\partial x^i} = \begin{cases} k_{\alpha_i}, & \text{if } i \ne 0; \\ \dfrac{\partial \phi_\alpha}{\partial x^0} = \dfrac{\partial \phi_\alpha}{\partial t} = \omega_\alpha, & \text{if } i = 0. \end{cases}$$

Using Gauges for the Neoclassical Economics. Identifying the utility with cumulate $\phi_\alpha(x,t)$ of the growth rates $\dot{\phi}_\alpha(x,\dot{x},t) = \nu_\alpha$ of the gross product Y in the industry α of (9), let's consider labor L, capital K and technological progress A as factor-productivities in a single industry model, with no direct relationship to time. Then the last equation in (9) can be rewritten as

$$\nu = k\dot{x}, \qquad \nu(x,\dot{x}) = (\bigtriangledown\phi, \dot{x}), \quad (10)$$

where $x = (L, K, A)$, and k is the vector of marginal utilities for factors L, K, A.

Cobb–Douglas Function. We assume $\bigtriangledown\phi = k = \left(\frac{\alpha}{L}, \frac{\beta}{K}, \frac{1}{A}\right)$ to expresses the contention that the factor-productivities L, K, A obey the Ricardian Law of Diminishing Returns [10–12]. Following Solow [1] the technological progress is assumed a residual A weighted to unity in regards to the growth rates of the gross product explained through factor-productivities of labor and capital. According to (10), we obtain Cobb–Douglas formula mathematically, whereas they were found empirically [13]. This points at a deeper meaning of this formula as perhaps representing the fundamental nature of the Law of Diminishing Returns, which was used to derive formula and Eq. (11)

$$\nu_Y(K,L,A) = \alpha\frac{\dot{L}}{L} + \beta\frac{\dot{K}}{K} + \frac{\dot{A}}{A}, \qquad \dot{Y} = \nu_Y Y, \qquad Y = AL^{\alpha}K^{\beta}. \tag{11}$$

The Solow model adds to the Cobb–Douglas function yet another equation, namely, the capital self-reproduction $\dot{K} = sY - \delta K$, from where we get $\nu_K(K,Y) = s\frac{Y}{K} - \delta$. The marginal utility of capital generation as a factor productivity $\frac{\partial\phi_K}{\partial Y} = s\frac{d(\ln K)}{d(\ln Y)}/\dot{K}$. The marginal self-correction of capital as a factor productivity $\frac{\partial\phi_K}{\partial K} = -\delta/\dot{K}$. Accordingly the Solow model can be rewritten as

$$\dot{Y} = \nu_Y Y, \qquad \dot{K} = \nu_K K. \tag{12}$$

This brings us to equations tracing the phase gradients for factor productivities.

Gauges in the Monetary Economy, the "Invisible Hand". *The Equations for Self-Regulating Markets.* We use equations based on (9), (10), where Y is the GDP, M is the monetary mass, which mutually interplay to achieve equilibrium

$$\dot{Y} = \nu_Y Y, \qquad \dot{M} = \nu_M M, \qquad \nu_Y = \frac{d\phi_Y(Y,M)}{dt}, \qquad \nu_M = \frac{d\phi_M(Y,M)}{dt}. \tag{13}$$

Similar to the case of neoclassics, we build full derivatives of phases ϕ_Y, ϕ_M, using partial derivatives as in the second equation of (10), i.e. $(\bigtriangledown\phi, \dot{x})$. Marginal utilities for factors Y and M are shown below [14].

$\frac{\partial\phi_Y}{\partial Y} = h\left(1 - \frac{Y}{CC}\right)/\dot{Y}$	Marginal self-generation of the gross product assumed equal to diminishing returns per unit of growth.
$\frac{\partial\phi_Y}{\partial M} = -i/\frac{\dot{M}}{M}$	Marginal correction of the gross product through the monetary mass, which are in inverse relationship to the latters growth rates.
$\frac{\partial\phi_M}{\partial Y} = \alpha/\frac{\dot{Y}}{Y}$	Marginal generation of the monetary mass through the supply of the real product is inversely proportional to its growth rates.
$\frac{\partial\phi_M}{\partial M} = -\varepsilon/\frac{\dot{M}}{M}$	Marginal self-correction of the monetary mass inversely proportional to the growth rates.

Substituting marginal utilities expressed through phases ϕ_Y, ϕ_M above into full derivatives v_Y, v_M, we obtain the equations of market's self-regulation for $\dot{\phi}_Y, \dot{\phi}_M$:

$$\dot{Y} = Y\left(h\left(1-\frac{Y}{CC}\right)-iM\right), \qquad \dot{M} = M(\alpha Y - \varepsilon M), \tag{14}$$

$$v_Y = h\left(1-\frac{Y}{CC}\right)-iM, \qquad v_M = \alpha Y - \varepsilon M. \tag{15}$$

Conclusion. By using gauges we demonstrated that self-regulation is a feature of a closed system [15, 16]. This creates a fundamental paradox since any closed system is devoid of sources of growth by definition.

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