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M a t h e m a t i c s

ON THE CONTINUITY OF EXTREMAL LENGTH

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In the paper the continuity of one conformal invariant extremal length is considered. A counterexample is constructed disproving the result of P.M. Tamrazov on the continuity of the extremal distance between two sets. Then some sufficient conditions for the continuity are given.

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Introduction. In the geometric functions theory an important role belongs to various conformal invariants. In particular, the extremal length of a curve family is an efficient tool for studying conformal and quasiconformal mappings. The present paper deals with one of the interesting aspects of theory: the continuity of the extremal length.

It is worth mentioning papers of F. Gehring [1], V. Wolontis [2], L. Ahlfors and A. Beurling [3] concerning this type of questions. Also, the continuity of the spatial condenser's conformal capacity has been proved in [4]. The conformal capacity of condensers, introduced in [5], is quite a general conformal invariant, coinciding with the module of the family of curves joining the plates of the condensers, which, in its turn, is the inverse of the extremal length of that family.

A more interesting result on the continuity of the extremal length belongs to P. M. Tamrazov [6]. In this work an assertion of Wolontis is disproved and one positive result is established concerning that question.

Below we will disprove the final statement of that work through a counterexample and will indicate some additional conditions sufficient for the result to remain true.

Let D be a domain in the complex plane; E and F are two separated sets in $D;\gamma$ is the family of all curves laying in D and joining E and F. The extremal length of γ is called extremal distance between E and F relative to D and is denoted by $\lambda_D(E,F)$.

The following result belongs to Wolontis.

Proposition 1. Let D be a domain; E and F are two separated compact subsets of D; $\{E_n,F_n\}$ is a sequence of pairs of compact subsets in D covering E and F and converging to E and F correspondingly (here convergence means that for any $\varepsilon > 0$ there exists a

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number N such that for any $n > N$ sets E_n and F_n lay in the ε-neighborhoods of E and F correspondingly). Then

$$\lim_{n\to\infty} \lambda_D(E_n, F_n) = \lambda_D(E, F). \tag{1}$$

In the proof of Proposition 1 the condition

$$E_n \supset E, F_n \supset F, n = 1, 2, \ldots, \tag{2}$$

is essential.

In [2] Wolontis states that the closedness requirement for the corresponding sets is not essential, that is, if $\overline{E}$ and $\overline{F}$ are the closures of E and F, then always $\lambda_D(\overline{E}, \overline{F}) = \lambda_D(E, F)$. This assertion was disproved by P.M. Tamrazov through a counterexample [6]. In the same paper, P.M. Tamrazov establishes a result, which, in some sense, is the generalization of Proposition 1 (Theorem 1). In the mentioned theorem condition (2) is removed and instead some supplementary metrical restrictors are introduced. Further, P.M. Tamrazov states an assertion about the role of the closedness feature of the corresponding point sets (Theorem 2). Next we bring these results.

Suppose

$$h(E_1, E_2) = \max\{ \sup_{z_1\in E_1} \inf_{z_2\in E_2} |z_1 - z_2|, \sup_{z_2\in E_2} \inf_{z_1\in E_1} \}. \tag{3}$$

$E_n \to E$ means that the sequence of sets $\{E_n\}$ converges to E in the metric h. It is easy to see that under the conditions of Proposition 1 $E_n \to E$ and $F_n \to F$. Suppose $d(E)$ is the lower bound of the diameters of the connected components of E.

Theorem 1. Let D be a domain; E and F are two separated compact subsets of D; $\{E_n \to F_n\}$ is a sequence of pairs of subsets in D satisfying

$$E_n \to E, F_n \to F. \tag{4}$$

If

$$\lim_{n\to\infty} d(E_n) > 0 \tag{5}$$

and

$$\lim_{n\to\infty} d(F_n) > 0 \tag{6}$$

then the equality (1) is true.

This Theorem remains true, if the condition (5) (analogously the condition (6)) is substituted by the requirement that the sets E_n (correspondingly F_n), $n = 1, 2, \ldots$, are connected.

Theorem 2. Let D be a domain; E and F are two separated sets with compact closures in D. If

$$d(E) > 0 \tag{7}$$

and

$$d(F) > 0, \tag{8}$$

then

$$\lambda_D(\overline{E}, \overline{F}) = \lambda_D(E, F). \tag{9}$$

Here also the Theorem remains true, if the condition (7) (analogously condition (8)) is substituted by the requirement that the set E (correspondingly F) is connected. Basing on these results, the author formulates a new Theorem.

Theorem 3. Let D be a domain; E and F are two separated subsets of D with compact closures in D; $\{E_n, F_n\}$ is a sequence of pairs of subsets in D satisfying (4). If the conditions (5) and (6) are fulfilled, then

$$\lim_{n\to\infty} \lambda_D(E_n, F_n) = \lambda_D(E, F) = \lambda_D(\overline{E}, \overline{F}). \tag{10}$$

This Theorem remains true also, if the condition (5) (analogously condition (6)) is substituted by the requirement that the sets E_n (correspondingly F_n), $n = 1, 2, \dots$, are connected.

Here the author silently assumes, that if $\lim\limits_{n\to\infty} d(E_n) > 0$, then automatically also $d(E) > 0$. But this is not true.

Below we give an example that disproves both Theorem 3 (in the formulation of the author) and the assertion of Wolontis on the equality .

Let D be a domain in R^2 containing the rectangle

$$\{x = (x_1, x_2) : 0 \leq x_1 \leq a, 0 \leq x_2 \leq b\}.$$

We denote by E the set of all the rational points of the lower base of the rectangle, and by $F-$ the set of all the rational points of the upper base. Then the closures of those sets $\overline{E}$ and $\overline{F}$ will obviously coincide with the lower and upper bases of the rectangle .

Further, suppose $E_n = \overline{E}$, $F_n = \overline{F}$, n = 1,2, . . .

Then the sets E_n and F_n are connected and, obviously, $E_n \to E$, $F_n \to F$ when $n \to \infty$.

Next we denote by Γ the family of all possible curves γ in D joining E and F, and by $\overline{\Gamma}-$ the family of all possible curves $\overline{\gamma}$ in D that join $\overline{E}$ and $\overline{F}$.

A nonnegative borelian function ρ is called a permissible metrics for the family Γ (denoted by $\rho\Lambda\Gamma$), if for every curve γ from Γ the inequality $\int_\gamma \rho dl \geq 1$ holds. We define the function ρ_0 to be equal $1/b$ inside the rectangle and to be zero anywhere else outside of it, and also let $\overline{\gamma}$ be an arbitrary curve from $\overline{\Gamma}$. We will show that ρ_0 is a permissible metrics for $\overline{\Gamma}$.

As

$$\int\limits_{\overline{\gamma}} \rho_0 dl = \frac{1}{b} \int\limits_{\overline{\gamma}} dl = \frac{1}{b} l(\overline{\gamma}) \geq \frac{1}{b} \cdot b = 1,$$

then $\rho_0 \Lambda \Gamma$, and we see that the set of all permissible metrics for $\overline{\Gamma}$ is nonempty. Now we compute $\int\int_{R^2} \rho^2 dx$ for ρ_0,

$$\int\limits_{R^2}\!\!\int \rho_0^2 dx = \frac{1}{b^2} \int\limits_{R^2}\!\!\int dx = \frac{1}{b^2} \cdot ab = \frac{a}{b}. \tag{11}$$

Next, if we take an arbitrary permissible metrics ρ for $\overline{\Gamma}$, then we will have

$$\int\limits_{R^2}\!\!\int \rho^2 dx_1 dx_2 \geq \int_0^a dx_1 \int_0^b \rho^2 dx_2 \geq \frac{1}{b} \int_0^a dx_1 \left(\int_0^b \rho dx_2 \right)^2. \tag{12}$$

Taking into account that the segment with length b joining $\overline{E}$ and $\overline{F}$ also belongs to the family $\overline{\Gamma}$, we notice that

$$\int_0^b \rho dx_2 \geq 1.$$

So, we get

$$\frac{1}{b} \int_0^a dx_1 \left(\int_0^b \rho dx_2 \right)^2 \geq \frac{1}{b} \int_0^a dx_1 = \frac{a}{b}.$$

Finally:

$$\int\limits_{R^2}\!\!\int \rho^2 dx \geq \frac{a}{b},$$

which means that ρ_0 is the extremal metrics and

$$M(\overline{\Gamma}) = \inf_{\rho} \int\limits_{R^2}\!\!\int dx = \int\limits_{R^2}\!\!\int \rho_0^2 dx = \frac{a}{b},$$

$$\lambda_D(\overline{E},\overline{F}) = \frac{1}{M(\overline{\Gamma})} = \frac{b}{a} < +\infty.$$

Now we consider the family Γ. We know that the module is an outer measure and, consequently, has the feature of semiadditivity. We also know that the family of all possible curves passing through the same fixed point of $\overline{R}^2$ is exceptional, that is it has a zero module. The set E consists of countable number of points. For each point x_i of E we will denote by Γ_i the family of curves from Γ passing through that point (x_i), $i = 1,2,\dots$. Then we will have $M(\Gamma_i) = 0$. As $\Gamma = \bigcup\limits_{i=1}^{+\infty} \Gamma_i$, then $M(\Gamma) = M\left(\bigcup\limits_{i=1}^{+\infty} \Gamma_i\right) \le \sum\limits_{i=1}^{+\infty} M(\Gamma_i) = 0$. Consequently, $M(\Gamma) = 0$, which means that $\lambda_D(E,F) = +\infty$. And this proves that $\lambda_D(\overline{E},\overline{F}) \ne \lambda_D(E,F)$, $\lim\limits_{n\to+\infty} \lambda_D(E_n,F_n) \ne \lambda_D(E,F)$.

We have proved that the statement of the Theorem 3 is not true. It will remain true, if we additionally impose the following conditions: $d(E) > 0$ and $d(F) > 0$. But in that case Theorem 3 will turn to be just the mechanical union of Theorems 1 and 2.

We notice also that in the counterexample constructed by P. M. Tamrazov in [6], if one puts $E_n = \overline{E}$, $F_n = \overline{F}$ for all n, then all the conditions of Theorem 3 will be fulfilled, but nevertheless the assertion of Theorem 3 will not hold true.

Dedicated to the memory of Promarz M. Tamrazov.

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