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M a t h e m a t i c s

ON SOLVABILITY OF A CLASS OF NONLINEAR INTEGRAL EQUATIONS WITH HAMMERSTEIN TYPE NONCOMPACT OPERATOR IN THE SPACE $L_1(R^+)$

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In the paper a class of nonlinear integral equations on the positive semi axis with noncompact Hammerstein type operator is studied.

The existence of nontrivial, nonnegative, integrable and bounded on R^+ solution is proved. Some specific examples of these equations representing independent interest are given.

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1. Introduction. Nonlinear integral equation of the form

$$\varphi(x) = \int_0^\infty K(x-t)G(t,\varphi(t))dt, \quad x \in R^+ \equiv [0,+\infty), \tag{1}$$

referred to the Hammerstein type equations have important applications in different areas of modern science. Particularly, one can meet similar species equation in the transfer theory, in the study of spectral lines, in the kinetic theory of gases, in econometrics, in cosmology, in solid mechanics, etc. (see [1–5]).

Importance of the positive solutions constructing (in various functional spaces) for such equations, in addition to purely theoretical interest, is that this class of equations is also used in above-mentioned areas of mathematical physics.

In [4–10], under different restrictions on the kernel K and the nonlinearity of G, existence of constructive theorems of positive (nonnegative and non-trivial) solutions are proved, as well as their asymptotic properties to $+\infty$ are investigated.

In this paper, using the earlier results [6] and making different assumptions about the function G, was obtained a possibility to construct a special cone-shaped segment leaving invariant the corresponding nonlinear Hammerstein operator, that

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allows to prove the existence of non-trivial, nonnegative global solution of (1) in the space of integrable and essentially bounded functions on R^+. After having proved the main Theorem, some examples of the function G, representing independent interest to illustrate the result, are given.

2. Notation and Auxiliary Facts. For the kernel K following conditions were assumed:

$$K(\tau) \geq 0, \quad \tau \in R, \quad \int\limits_{-\infty}^{+\infty} K(\tau)d\tau = 1, \quad K \in L_1(R) \cap L_\infty(R), \tag{2}$$

$$\nu(K) \equiv \int\limits_{-\infty}^{+\infty} \tau K(\tau)d\tau < 0, \tag{3}$$

the absolute convergence of the last integral was assumed also.

Let E be one of the following Banach spaces:

$$L_p(R^+), \quad 1 \leq p \leq +\infty, \; C_M(R^+), \quad C_0(R^+),$$

where $C_M(R^+)$ is the space of continuous and bounded functions on R^+ and $C_0(R^+)$ is a space of continuous on R^+ functions vanishing at infinity.

Let denote by $\mathcal{K}$ the integral operator named by Wiener–Hopf generated with kernel K:

$$(\mathcal{K}f)(x) = \int\limits_0^{+\infty} K(x-t)f(t)dt, \quad f \in E. \tag{4}$$

It is known, that the operator $I - \mathcal{K}$ permits Volterra factorization (see [11])

$$I - \mathcal{K} = (I - V_-)(I - V_+), \tag{5}$$

where I is the identity operator and $V_\pm$ are upper and lower Volterra operators of the form

$$(V_+f)(x) = \int\limits_0^x v_+(x-t)f(t)dt, \quad (V_-f)(x) = \int\limits_x^\infty v_-(t-x)f(t)dt, \quad f \in E. \tag{6}$$

Kernels $v_\pm \in L_1(R^+)$ are nonnegative and are determined from the system of nonlinear factorization equations of N. B. Engibaryan:

$$v_\pm(x) = K(\pm x) + \int\limits_0^x v_\pm(x+t)v_\mp(t)dt, \quad x \in R^+, \tag{7}$$

and have the following properties:

$$\gamma_+ \equiv \int\limits_0^\infty v_+(x)dx < 1, \quad \gamma_- \equiv \int\limits_0^\infty v_-(x)dx = 1. \tag{8}$$

Let the $\omega(t,u)$ be a function defined on $R^+ \times R$ and satisfying the following conditions:

a) there exists a number $A \geq 0$ such that for any fixed $t \in R^+$ the function $\omega(t,u)$ is monotonically decreasing with respect to u on the set $[A,\infty)$;

b) the function $\omega(t,u)$ is satisfying the Caratheodory's condition with respect to the argument u on the set $\Omega_A \equiv R^+ \times [A,\infty)$, i.e. for each fixed $u \in [A,\infty)$ the function $\omega(t,u)$ is measurable in t and for almost all $t \in R^+$ the function $\omega(t,u)$ is continuous when $u \in [A,+\infty)$. This condition we will write further, in brief, in the following form: $\omega \in Car_u(\Omega_A)$;

c) there exists a measurable function

$$\omega^0 \in L_1(R^+) \cap C_0(R^+), \;\; m_1(\omega^0) \equiv \int_0^\infty t\omega^0(t)dt < +\infty,$$

defined on R^+ such that $\omega^0(t) \downarrow$ on $[A,+\infty)$ and

$$0 \leq \omega(t,u) \leq \omega^0(t+u), \;\; (t,u) \in \Omega_A. \tag{9}$$

Consider the following nonlinear equation of Hammerstein type

$$f(x) = \int_0^\infty K(x-t)(f(t) - \omega(t,f(t))dt, \;\; x \in R^+, \tag{10}$$

with respect to the unknown function $f(x)$. Here is assumed that the kernel K and the function $\omega(t,u)$ are satisfying the conditions (2), (3), and a) – c).

Then from the results of [6] follows, that (10) has the one-parameter family of positive solutions $\left\{f_\gamma(x)\right\}_{\gamma\in\Delta}$ with the following properties:

- $f_\gamma \in L_\infty(R^+), \;\; \gamma \in \Delta$;
- $\lim\limits_{x\to+\infty} f_\gamma(x) = \dfrac{2\gamma}{1-\gamma_+}, \;\; \forall\gamma \in \Delta$;
- if $\gamma_1, \gamma_2 \in \Delta$ and $\gamma_1 > \gamma_2$, then $f_{\gamma_1}(x) - f_{\gamma_2}(x) \geq 2(\gamma_1 - \gamma_2), \;\; x \in R^+$;
- the following two-sided estimation for each of these solutions is true:

$$S_\gamma(x) \leq 2S_\gamma(x) - Q(x) \leq f_\gamma(x) \leq 2S_\gamma(x), \;\;\; x \in R^+, \;\; \gamma \in \Delta, \tag{11}$$

where $Q(x)$ is the unique positive solution of the following inhomogeneous Wiener–Hopf equation:

$$Q(x) = 2\omega^0(x+A) + \int_0^\infty K(x-t)Q(t)dt, \;\; x \in R^+, \tag{12}$$

having the properties

$$Q(x) \in L_1(R^+) \cap L_\infty(R^+), \;\; \lim_{x\to+\infty} Q(x) = 0, \tag{13}$$

and $S_\gamma(x) = \gamma S(x)$, where $S(x)$ is the unique solution of the following initial value problem for an homogeneous Wiener–Hopf equation:

$$\begin{cases} S(x) = \displaystyle\int_0^\infty K(x-t)S(t)dt, \;\; x \in R^+, \\ S(0) = 1, \end{cases} \tag{14}$$

has the following properties:

$$S(x)\uparrow \text{with respect to } x \text{ on } R^+,\ \lim_{x\to+\infty} S(x)=\frac{1}{1-\gamma_+}. \tag{15}$$

Range of parameters Δ is given by the following formula

$$\Delta\equiv[\max(\kappa,\gamma_0),+\infty), \tag{16}$$

where $\kappa\equiv\sup\limits_{x\in R^+} Q(x)$ and $\gamma_0\geq A$ is a fixed number, for which inequality is fulfilled at first $\omega^0(\gamma_0)<\gamma_0$ (due to the properties of the functions ω^0 such a number exists).

3. The Main Results.

Theorem. Let $G(t,u)$ be a real function which is defined on $R^+\times R$ and is satisfying the following conditions, there exists a number

$$\eta\geq\frac{2\max(\kappa,\gamma_0)}{1-\gamma_+} \tag{17}$$

such that:

1) for any fixed $t\in R^+$ the function $G(t,u)$ is increasing with respect to u on $[g(t),\eta-A]$, where

$$g(t)\equiv\int_0^\infty K(t-\tau)\omega(\tau,\eta)d\tau,\ \ t\in R^+, \tag{18}$$

and ω satisfies the conditions a)–c);

2) $G\in Car_u(\prod_\eta)$, where $\prod_\eta\equiv R^+\times[0,\eta-A]$;

3) two-sided estimation is true:

$$u+\omega(t,\eta)\leq G(t,u)\leq u+\omega(t,\eta-u) \tag{19}$$

for $t\in R^+,\ \ u\in[g(t),\eta-A]$.

Then Eq. (1), under conditions of Eq. (2), (3), has a nonnegative, non-trivial solution in the space of integrable and essentially bounded functions on R^+. Moreover, this solution is vanishing at infinity.

Proof. The results that are given in part 2 of present paper imply, that since the number η satisfies (17), the Eq. (10) has a positive essential bounded solution $f_{\gamma^*}(x),\ \ \gamma^*=\dfrac{\eta(1-\gamma_+)}{2}\in\Delta$ and $\lim\limits_{x\to+\infty} f_{\gamma^*}(x)=\eta$, moreover,

$$S_{\gamma^*}(x)\leq f_{\gamma^*}(x)\leq\eta,\ \ 2S_{\gamma^*}(x)-Q(x)\leq f_{\gamma^*}(x)\leq 2S_{\gamma^*}(x),\quad x\in R^+. \tag{20}$$

For the correctness, we prove that

$$g(x)\leq\eta-A,\ \ x\in R^+. \tag{21}$$

Really, taking into account the properties of the functions $f_{\gamma^*}(x)$, we have

$$g(x)=\int_0^\infty K(x-t)\omega(t,\eta)dt\leq\int_0^\infty K(x-t)\omega(t,f_{\gamma^*}(t))dt=$$

$$=\int_0^\infty K(x-t)f_{\gamma^*}(t)dt-f_{\gamma^*}(x)\leq\eta\int_0^\infty K(x-t)dt-A\leq\eta-A$$

as

$$\eta \geq \frac{2\max(\kappa,\gamma_0)}{1-\gamma_+} \geq \max(\kappa,\gamma_0) \geq \gamma_0 \geq A,$$

$$f_{\gamma^*}(x) \geq S_{\gamma^*}(x) \geq \gamma^* \geq \gamma_0 \geq A,$$

$\omega(t,u) \downarrow$ with respect to u on $[A,+\infty)$, where the kernel is satisfying (2).

Below we verify that $\eta - f_{\gamma^*} \in L_1(R^+)$. So, from (20) follows that

$$0 \leq \eta - f_{\gamma^*} \leq \eta - 2S_{\gamma^*}(x) + Q(x) = 2\gamma^*\Big(\frac{1}{1-\gamma_+} - S(x)\Big) + Q(x). \tag{22}$$

Since $Q \in L_1(R^+) \cap L_\infty(R^+)$, in [5] was proved that $\frac{1}{1-\gamma_+} - S \in L_1(R^+)$, than from (22) we obtain the required inclusion.

Along with Eq. (1), we consider the following auxiliary equation of Wiener–Hopf:

$$\phi(x) = g(x) + \int_0^\infty K(x-t)\phi(t)dt, \;\; x \in R^+, \tag{23}$$

with respect to the unknown function $\phi(x)$, where $g(x)$ is given by (18).

Let consider the following iteration:

$$\phi_{n+1}(x) = g(x) + \int_0^\infty K(x-t)\phi_n(t)dt, \;\; \phi_0(x) \equiv 0, \;\; n = 0,1,2..., \;\; x \in R^+. \tag{24}$$

Using induction method, it is easy to check that

$$\phi_n(x) \uparrow \text{with respect to } \; n. \tag{25}$$

We will prove that

$$\phi_n(x) \leq \eta - f_{\gamma^*}(x), \;\; n = 0,1,2..., \;\; x \in R^+. \tag{26}$$

In the case when $n = 0$ inequality (26) follows from (20). Suppose that it is true for some $n \in \mathbb{N}$ and prove it for $n+1$. Using (18) and the properties of the function $\omega(t,u)$, from (24) we have:

$$\phi_{n+1}(x) \leq \int_0^\infty K(x-t)\omega(t,\eta)dt + \int_0^\infty K(x-t)(\eta - f_{\gamma^*}(t))dt =$$

$$= \int_0^\infty K(x-t)\omega(t,\eta)dt + \eta\int_{-\infty}^x K(\tau)d\tau - f_{\gamma^*}(x) - \int_0^\infty K(x-t)\omega(t,f_{\gamma^*}(t))dt \leq$$

$$\leq \eta - f_{\gamma^*}(t) - \int_0^\infty K(x-t)(\omega(t,f_{\gamma^*}(t)) - \omega(t,\eta)dt \leq \eta - f_{\gamma^*}(t).$$

Therefore, the sequence of functions $\{\phi_n(x)\}_{n=0}^\infty$ has a pointwise limit, while $n \to \infty$: $\lim\limits_{n\to\infty}\phi_n(x) = \phi(x)$, where the function $\phi(x)$, according to the Lebesgue theorem, satisfies the inequality (23). From (25) and (26) it follows also, that

$$g(x) \leq \phi(x) \leq \eta - f_{\gamma^*}(x), \;\; x \in R^+. \tag{27}$$

Let come back to the solution of the initial Eq. (1). Let consider the following special iteration for the main nonlinear Eq. (1):

$$\varphi_{n+1}(x)=\int\limits_0^\infty K(x-t)G(t,\varphi_n(t))dt,\ \ \varphi_0(x)=\phi(x),\ \ x\in R^+. \tag{28}$$

Applying induction method on n, we prove that

$$\varphi_n(x)\uparrow \text{ whit respect to } \ n,\ \ \varphi_n(x)\le\eta-f_{\gamma^*}(x),\ n=0,1,2...,x\in R^+. \tag{29}$$

In the case, when $n=0$ inequality (29) follows from (27).

Now let verify

$$\varphi_1(x)\ge\varphi_0(x),\ \ x\in R^+.$$

Taking into consideration (27), (19), the monotonicity $G(t,u)$ with respect to u on $[g(t),\eta-A]$, from (28) we have

$$\varphi_1(x)\ge\int\limits_0^\infty K(x-t)(\varphi_0(t)+\omega(t,\eta))dt=\int\limits_0^\infty K(x-t)\phi(t)dt+g(x)=\phi(x)=\varphi_0(x).$$

Assume that

$$\varphi_n(x)\ge\varphi_{n-1}(x)\ \text{ and }\ \varphi_n(x)\le\eta-f_{\gamma^*}(x),\ \ x\in R^+,$$

for some $n\in\mathbb{N}$. Then, again using (19), (27) and the monotonicity $G(t,u)$ with respect to u, from (28) we obtain

$$\varphi_{n+1}(x)\le\int\limits_0^\infty K(x-t)G(t,\eta-f_{\gamma^*}(t))dt\le\int\limits_0^\infty K(x-t)(\eta-f_{\gamma^*}(t)+\omega(t,f_{\gamma^*}(t)))dt\le$$

$$\le\eta-\int\limits_0^\infty K(x-t)(f_{\gamma^*}(t)-\omega(t,f_{\gamma^*}(t)))dt=\eta-f_{\gamma^*}(x)$$

and

$$\varphi_{n+1}(x)\ge\int\limits_0^\infty K(x-t)G(t,\varphi_{n-1}(t))dt=\varphi_n(x).$$

Thus, the sequence of functions $\{\varphi_n(x)\}_{n=0}^\infty$ has a pointwise limit: $\lim\limits_{n\to\infty}\varphi_n(x)=\varphi(x)$. From condition 2), by the B. Levi's theorem (see [12]), we conclude that $\varphi(x)$ satisfies Eq. (1), and due to (29) the following inequalities chain is true:

$$0\le\phi(x)\le\varphi(x)\le\eta-f_{\gamma^*}(x),\ \ x\in R^+. \tag{30}$$

Since $\eta-f_{\gamma^*}\in L_1(R^+)\cap L_\infty(R^+)$, $\lim\limits_{x\to+\infty}f_{\gamma^*}(x)=\eta$, then from Eq. (30) it follows that $\varphi\in L_1(R^+)\cap L_\infty(R^+)$ and $\lim\limits_{x\to+\infty}\varphi(x)=0$, $\varphi(x)\not\equiv0$. $\square$

4. Examples of $G(t,u)$ Functions. To illustrate the result, we give examples of function $G(t,u)$:

i) $G(t,u)=\sqrt{(u+\omega(t,\eta))(u+\omega(t,\eta-u))}$;

j) $G(t,u)=\dfrac{2(u+\omega(t,\eta))(u+\omega(t,\eta-u))}{2u+\omega(t,\eta)+\omega(t,\eta-u)}$;

k) $G(t,u)=u+\dfrac{\omega(t,\eta)u}{u+\alpha(t)}-\omega(t,\eta)+\omega(t,\eta-u)$,

where $\alpha(t)$ is a measurable function defined on R^+ and satisfies the inequality

$$0\leq\alpha(t)\leq\frac{g(t)(\omega(t,\eta-g(t))-\omega(t,\eta))}{2\omega(t,\eta)},\ \ t\in R^+. \tag{31}$$

For examples i) and j) conditions of the Theorem are easily verifiable. Dwell on the example k). Note, that $G(t,u)\uparrow$ with respect to u on $[g(t),\eta-A]$, if $\omega(t,u)\downarrow$ with respect to u on $[A,+\infty)$, because of $\dfrac{u}{u+\alpha(t)}\uparrow$ with respect to u. On the other hand, for example k) function $G\in Car_u(\prod_\eta)$, because of $\omega\in Car_u(R^+\times[0,\eta-A])$.

Inequality $G(t,u)\leq u+\omega(t,\eta-u)$ is obvious, since $\alpha(t)\geq 0$. We will prove that $G(t,u)\geq u+\omega(t,\eta)$ for $u\geq g(t),\ \ t\in R^+$. The last inequality is equivalent to inequality

$$\rho(t,u)\equiv\frac{\omega(t,\eta)u}{u+\alpha(t)}+\omega(t,\eta-u)\geq 2\omega(t,\eta),\ \ u\geq g(t),t\in R^+.$$

Since $\rho(t,u)\ \uparrow$ with respect to u on $[g(t),\eta-A]$, then by (31) we have

$$\rho(t,u)\geq\frac{\omega(t,\eta)g(t)}{g(t)+\alpha(t)}+\omega(t,\eta-g(t))\geq$$

$$\geq\frac{2\omega^2(t,\eta)g(t)}{\omega(t,\eta)g(t)+g(t)\omega(t,\eta-g(t))}+\omega(t,\eta-g(t))=$$

$$=\frac{2\omega^2(t,\eta)+\omega(t,\eta)\omega(t,\eta-g(t))+\omega^2(t,\eta-g(t))}{\omega(t,\eta)+\omega(t,\eta-g(t))}=$$

$$=\omega(t,\eta)+\frac{\omega^2(t,\eta)+\omega^2(t,\eta-g(t))}{\omega(t,\eta)+\omega(t,\eta-g(t))}\geq$$

$$\geq\omega(t,\eta)+\frac{\omega^2(t,\eta)+\omega(t,\eta-g(t))\omega(t,\eta)}{\omega(t,\eta)+\omega(t,\eta-g(t))}=2\omega(t,\eta).$$

Note that in contrast to the examples i) and j), in the example k) the function $G(t,u)$ has the property of criticality

$$G(t,0)\equiv 0,\ \ t\in R^+,$$

i.e. equation

$$\varphi(x)=\int_0^\infty K(x-t)\Big[\varphi(t)+\frac{\omega(t,\eta)\varphi(t)}{\varphi(t)+\alpha(t)}+\omega(t,\eta-\varphi(t))-\omega(t,\eta)\Big]dt,\ \ x\in R^+, \tag{32}$$

has the trivial solution $\varphi(x) \equiv 0$. From the Theorem it follows that Eq. (32) except the trivial solution has also nonnegative non-trivial solution

$$\varphi : \varphi \in L_1(R^+) \cap L_\infty(R^+), \lim_{x\to+\infty} \varphi(x) = 0, \phi(x) \le \varphi(x) \le \eta - f_{\gamma^*}(x),\ x \in R^+.$$

At the end we also give an example of the function $\omega(t,u)$:

$$\omega(t,u) = ue^{-\alpha_0(t+u)},$$

where $\alpha_0 > 0$ is an arbitrary constant. Here we note that $\omega(t,u) \downarrow$ with respect to u on $\left[\frac{1}{\alpha_0}, +\infty\right)$ and obviously satisfies the conditions a) – c).

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REFERENCES

1. **Engibaryan N.B.** On a Problem in Nonlinear Radiative Transfer. // Astrophysics, 1996, v. 2, № 1, p 12–14.
2. **Engibaryan N.B., Khachatryan A.Kh.** On Nonlinear Theory Dynamics of Dilute Gas. // Matem. Mod., 2004, v. 16, p. 67–74 (in Russian).
3. **Khachatryan A.Kh., Khachatryan Kh.A.** Qualitative Difference Between Solutions for a Model Boltzmann Equation in the Linear and Nonlinear Cases. // Theoret. and Math. Phys., 2012, v. 172, № 3, p. 1315–1320.
4. **Aref'eva I.Y.** Nonlocal String Tachyon as a Model For Cosmological Dark Energy. // Int. Conf. Proc., 2006, v. 826, p. 301–311.
5. **Khachatryan A., Khachatryan Kh.** On Solvability of a Nonlinear Problem in Theory of Income Distribution. // Eurasian Math. Journal, 2011, v. 2, № 2, p. 75–88.
6. **Khachatryan Kh.A.** On a Class of Nonlinear Integral Equations with a Noncompact Operator. // Journal of Contemporary Math. Analysis, 2011, v. 46, № 2, p. 89–100.
7. **Vladimirov V.S.** The Equation of p-Adic Closed String for the Scalar Tachyon Field. // Science in China. Ser.A., Mathematics, 2008, v. 51, № 4, p. 754–764.
8. **Arabadzhyan L.G.** Solution of an Integral Equation of Hammerstein Type. // Izvestia NAN Armenii, 1997, v. 32, № 1, p. 21–28 (in Russian).
9. **Khachatryan Kh.A.** On a Class of Integral Equations of Urysohn Type with Strong Non-Linearity. // Izv. RAN. Ser. Mat., 2012, v. 76, № 1, p. 173–200 (in Russian).
10. **Khachatryan A.Kh., Khachatryan Kh.A.** On Solvability of One Class of Hammerstein Nonlinear Integral Equations. // Bul. Acad., Ştiinţe Repub. Moldova, 2010, № 2, p. 67–83.
11. **Arabadzhyan L.G., Engibaryan N.B.** Convolution Equations and Nonlinear Functional Equations. // Itogi Nauki i Tekhn., Ser. Mat. Anal., 1984, v. 22, p. 175–244 (in Russian).
12. **Kolmogorov A.N., Fomin S.V.** Elements of the Theory of Functions and Functional Analysis. M.: Nauka, 1981, 544 p. (in Russian).