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M a t h e m a t i c s

ON THE MINIMAL NUMBER OF NODES UNIQUELY DETERMINING ALGEBRAIC CURVES

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It is well-known that the number of n-independent nodes determining uniquely the curve of degree n passing through them equals to $N-1$, where $N=\frac{1}{2}(n+1)(n+2)$. It was proved in [1], that the minimal number of n-independent nodes determining uniquely the curve of degree $n-1$ equals to $N-4$. The paper also posed a conjecture concerning the analogous problem for general degree $k \le n$. In the present paper the conjecture is proved, establishing that the minimal number of n-independent nodes determining uniquely the curve of degree $k \le n$ equals to $\frac{(k-1)(2n+4-k)}{2}+2$.

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Introduction. Denote the space of all bivariate polynomials of total degree $\le n$ by Π_n:

$$\Pi_n = \left\{ \sum_{i+j\le n} a_{ij}x^iy^j \right\}.$$

We have

$$N := N_n := \dim \Pi_n = \binom{n+2}{2}.$$

Consider a set of s distinct nodes

$$\mathcal{X}_s = \{(x_1,y_1),(x_2,y_2),\dots,(x_s,y_s)\}.$$

The problem of finding a polynomial $p \in \Pi_n$, which satisfies the conditions

$$p(x_i,y_i) = c_i, \quad i = 1,\dots,s, \tag{1}$$

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is called interpolation problem.

A polynomial $p \in \Pi_n$ is called an n-fundamental polynomial for a node $A = (x_k, y_k) \in \mathcal{X}_s$ if

$$p(x_i, y_i) = \delta_{ik},\ i = 1, \ldots, s,$$

where δ is the Kronecker symbol. We denote this fundamental polynomial by $p_k^\star = p_A^\star = p_{A,\mathcal{X}_s}^\star$. Sometimes we call fundamental also a polynomial that vanishes at all the nodes of $\mathcal{X}_s$, but one, since it is a nonzero constant times a fundamental polynomial.

Next, let us consider an important concept of n-independence (see [2, 3]).

Definition 1. A set of nodes $\mathcal{X}$ is called *n-independent*, if all its nodes have n-fundamental polynomials. Otherwise, if a node has no n-fundamental polynomial, then $\mathcal{X}$ is called *n-dependent*.

Fundamental polynomials are linearly independent. Therefore, a necessary condition of n-independence of $\mathcal{X}_s$ is $s \leq N$.

Suppose a node set $\mathcal{X}_s$ is n-independent. Then, by the Lagrange formula, we obtain a polynomial $p \in \Pi_n$ satisfying the interpolation conditions (1):

$$p = \sum_{i=1}^{s} c_i p_i^\star.$$

In view of this we readily get that the node set $\mathcal{X}_s$ is n-independent if and only if the interpolating problem (1) is *solvable*, that means for any data $(c_1, \ldots, c_s)$ there is a polynomial $p \in \Pi_n$ (not necessarily unique) satisfying the interpolation conditions (1).

Definition 2. The interpolation problem with a set of nodes $\mathcal{X}_s$ and Π_n is called *n-poised*, if for any data $(c_1, \ldots, c_s)$, there is a *unique* polynomial $p \in \Pi_n$ satisfying the interpolation conditions (1).

A necessary condition of n-poisedness of $\mathcal{X}_s$ is $s = N$.

For node sets of cardinality N we have the following

Proposition 1. A set of nodes $\mathcal{X}_N$ is n-poised, if and only if

$$p \in \Pi_n \text{ and } p\big|_{\mathcal{X}_N} = 0 \quad \Longrightarrow \quad p = 0.$$

Thus $\mathcal{X}_N$ is n-poised if and only if it is n-independent.

Evidently, any subset of n-poised set is n-independent. According to the next lemma, any n-independent set is a subset of some n-poised set (see, e.g., [4], Lemma 2.1).

Lemma 1. Any n-independent set $\mathcal{X}_s$ with $s < N$ can be extended to a n-poised set.

Below a well-known construction of n-poised set is described (see [5, 6]).

Definition 3. A set of $N = 1 + \cdots + (n+1)$ nodes is called Berzolari–Radon set for degree n or briefly BR_n set, if there exist lines $l_1, l_2, \ldots, l_{n+1}$ such that the sets $l_1,\ l_2 \setminus l_1,\ l_3 \setminus (l_1 \cup l_2), \ldots,\ l_{n+1} \setminus (l_1 \cup \cdots \cup l_n)$ contain exactly $(n+1), n, n-1, \ldots, 1$ nodes respectively.

Algebraic curve in plane is the zero set of some bivariate polynomial of degree at least 1. The same letter, say p, is used to denote the polynomial $p \in \Pi_k \setminus \Pi_{k-1}$ and the corresponding curve p of degree k defined by the equation $p(x,y) = 0$.

According to the following well-known statement, there are no more than $n+1$ number of n-independent points in any line.

Proposition 2. Assume that l is a line and $\mathcal{X}_{n+1}$ is any subset of l containing $n+1$ points. Then we have that

$$p \in \Pi_n \quad \text{and} \quad p|_{\mathcal{X}_{n+1}} = 0 \Rightarrow \quad p = lr, \text{ where } r \in \Pi_{n-1}.$$

Denote

$$d := d(n,k) := N_n - N_{n-k} = k(2n+3-k)/2.$$

The following is a generalization of Proposition 2.

Proposition 3. ([7], Prop. 3.1). Let q be an algebraic curve of degree $k \le n$ without multiple components. Then we have:

i) any subset of q containing more than $d(n,k)$ nodes is n-dependent;

ii) any subset $\mathcal{X}_d$ of q containing exactly $d(n,k)$ nodes is n-independent if and only if the following condition holds:

$$p \in \Pi_n \quad \text{and} \quad p|_{\mathcal{X}_d} = 0 \Rightarrow p = qr, \text{ where } r \in \Pi_{n-k}.$$

Suppose that $\mathcal{X}$ is an n-poised set of nodes and q is an algebraic curve of degree $k \le n$. Then, of course, any subset of $\mathcal{X}$ is n-independent, too. Therefore, according to Proposition 3, i), at most $d(n,k)$ nodes of $\mathcal{X}$ can lie on the curve q. Let us mention that a special case of this when q is a set of k lines is proved in [8].

This motivates the following definition (see [7], Def. 3.1).

Definition 4. Given an n-independent set of nodes $\mathcal{X}_s$ with $s \ge d(n,k)$. A curve of degree $k \le n$ passing through $d(n,k)$ points of $\mathcal{X}_s$ is called maximal for $\mathcal{X}_s$.

In view of Propositions 2 and 3, any set of $n+1$ nodes located in a line is n-independent. Note that a maximal line, as a line passing through $n+1$ nodes, is defined in [9].

The following lemmas (see [3], Proposition 1.10, Lemma 2.2) will be needed in the sequel.

Lemma 2. The following two conditions are equivalent:

i) there is a k-poised subset of a set $\mathcal{X}$;

ii) there is no algebraic curve of degree k passing through all the points of $\mathcal{X}$.

Lemma 3. Suppose that a node set $\mathcal{X}$ is n-independent and a node $A \notin \mathcal{X}$ has a n-fundamental polynomial with respect to the set $\mathcal{X} \cup \{A\}$. Then the last node set is n-independent too.

Denote the linear space of polynomials of total degree $\le n$ vanishing on $\mathcal{X}$ by

$$\mathcal{P}_{n,\mathcal{X}} = \left\{ p \in \Pi_n : p\big|_{\mathcal{X}} = 0 \right\}.$$

The following is well-known (see, e.g., [3]).

Proposition 4. For any node set $\mathcal{X}$ we have

$$\dim \mathcal{P}_{n,\mathcal{X}} \ge N - \#\mathcal{X}.$$

Moreover, equality takes place here if and only if the set $\mathcal{X}$ is n-independent.

From here one can readily get (see [10], Corollary 2.4).

Corollary 1. Let $\mathcal{Y}$ be a maximal n-independent subset of $\mathcal{X}$, i.e., $\mathcal{Y} \subset \mathcal{X}$ is n-independent and $\mathcal{Y} \cup \{A\}$ is n-dependent for any $A \in \mathcal{X} \setminus \mathcal{Y}$. Then we have that

$$\mathcal{P}_{n,\mathcal{Y}} = \mathcal{P}_{n,\mathcal{X}}. \tag{2}$$

Proof. We have $\mathcal{P}_{n,\mathcal{X}} \subset \mathcal{P}_{n,\mathcal{Y}}$, since $\mathcal{Y} \subset \mathcal{X}$. Now suppose $p \in \Pi_n$, $p\big|_{\mathcal{Y}} = 0$ and A is any node of $\mathcal{X}$, we will get that $\mathcal{Y} \cup \{A\}$ is dependent and, therefore, in view of Lemma 3, we get $p\big|_A = 0$. □

From (2) and Proposition 4 (part "moreover"), we have

$$\dim \mathcal{P}_{n,\mathcal{X}} = N - \#\mathcal{Y}, \tag{3}$$

where $\mathcal{Y}$ is any maximal n-independent subset of $\mathcal{X}$. Thus all the maximal n-independent subsets of $\mathcal{X}$ have the same cardinality, which is called *the Hilbert n-function* of $\mathcal{X}$ and is denoted by $\mathcal{H}_n(\mathcal{X})$. Hence, according to (3), we have

$$\dim \mathcal{P}_{n,\mathcal{X}} = N - \mathcal{H}_n(\mathcal{X}).$$

Proposition 5. Assume that σ is an algebraic curve of degree k without any multiple component and $\mathcal{X}_s \subset \sigma$ is an arbitrary set of s n-independent points with $s < d(n,k)$. Then the set $\mathcal{X}_s$ can be extended to a maximal n-independent set $\mathcal{X}_d \subset \sigma$, where $d = d(n,k)$.

Proof. It suffices to show that there is a point $A \in \sigma$ such that the set $\mathcal{X}_{s+1} := \mathcal{X}_s \cup \{A\}$ is n-independent. Assume to the contrary that there is no such point, i.e. the set $\mathcal{X}_{s+1} := \mathcal{X}_s \cup \{A\}$ is n-dependent for any $A \in \sigma$. Then, in view of Lemma 3, A has no fundamental polynomial with respect to the set $\mathcal{X}_{s+1}$. In other words, we have

$$p \in \Pi_n \text{ and } p\big|_{\mathcal{X}_s} = 0 \quad \Longrightarrow \quad p(A) = 0 \text{ for any } A \in \sigma.$$

From here we obtain that

$$\mathcal{P}_{n,\mathcal{X}_s} \subset \mathcal{P}_{n,\sigma} := \{q\sigma : q \in \Pi_{n-k}\}.$$

Now, in view of Proposition 4, from here we get

$$N - s = \dim \mathcal{P}_{n,\mathcal{X}_s} \le \dim \mathcal{P}_{n,\sigma} = N_{n-k}.$$

Therefore, $s \ge d(n,k)$, which contradicts the hypothesis of Proposition. □

The Main Result. Below we determine the minimal number of n-independent nodes that uniquely determine the curve of degree k, $k \le n$, passing through them.

Theorem 1. Assume that $\mathcal{X}$ is an arbitrary set of $(d(n,k-1)+2)$ n-independent nodes lying on a curve of degree k with $k \le n$. Then the curve is determined uniquely. Moreover, there is a set $\mathcal{X}_1$ of $(d(n,k-1)+1)$ n-independent nodes, such that more than one curves of degree k pass through all its nodes.

Proof. Let us start with the part "moreover". Consider the part of Berzolari–Radon set BR_n belonging to the first $k-1$ lines $\ell_1, \dots, \ell_{k-1}$, i.e.

$$\mathcal{X}_0 = BR_n \cap [\ell_1 \cup \dots \cup \ell_{k-1}].$$

We have that the set $\mathcal{X}_0$ consists of $d(n,k-1) = (n+1) + n + (n-1) + \dots + (n-k+3)$ nodes. We get a desired set $\mathcal{X}_1$ by adding to this set a node $A \in BR_n \setminus \mathcal{X}_0$, i.e.

$\mathcal{X}_1 := \mathcal{X}_0 \cup \{A\}$. Now we have that the set $\mathcal{X}_1$ is n-independent, since it is a subset of n-poised set BR_n and $\#\mathcal{X}_1 = d(n,k-1)+1$. Finally, consider the curves of degree k of the form ℓq_{k-1}, where ℓ is any line passing through A and $q_{k-1} = \ell_1 \cdots \ell_{k-1}$. It remains to notice that all these curves of degree k pass through all the nodes of $\mathcal{X}_1$.

Now let us prove the first statement of Theorem. Assume the converse that there are two curves $\sigma, \sigma' \in \Pi_k$, which pass through all the $d(n,k-1)+2$ nodes of $\mathcal{X}$. In view of Proposition 5, let us enlarge $\mathcal{X}$ to a set $\bar{\mathcal{X}} \subset \sigma$ of $d(n,k)$ n-independent nodes, by adding $n-k$ $[= d(n,k) - (d(n,k-1)+2)]$ nodes $A_1, \dots, A_{n-k} \in \sigma$, i.e. $\bar{\mathcal{X}} = \mathcal{X} \cup \{A_i\}_{i=1}^{n-k}$. Then we obtain $d(n,k)$ n-independent nodes in σ and, therefore, this curve becomes a maximal curve of degree k with respect to the set $\bar{\mathcal{X}}$.

Next let us choose $n-k$ distinct lines $l_1, \dots, l_{n-k}$, which pass through the points $A_1, \dots, A_{n-k}$ respectively, and are not components (factors) of σ.

Set the polynomial

$$p = \sigma' \ell_1 \dots \ell_{n-k} \in \Pi_n.$$

Notice that p vanishes at all $d(n,k)$ n-independent points of $\bar{\mathcal{X}}$. Therefore, by the Proposition 3, ii), it has the following form

$$p = \sigma q, \quad q \in \Pi_{n-k}.$$

Thus, we have

$$\sigma' \ell_1 \dots \ell_{n-k} = \sigma q. \tag{4}$$

The lines $\ell_1, \dots, \ell_{n-k}$ are not factors of σ, so they are factors of $q \in \Pi_{n-k}$, which means that $q = c\ell_1 \dots \ell_{n-k}$, where $c \neq 0$. Consequently we get from (4) that

$$\sigma' = c\sigma,$$

or in other words the curves σ' and σ coincide. □

Now let present two corollaries of Theorem. The first one concerns an arbitrary n-independent set $\mathcal{X}$ with $\#\mathcal{X} \geq d(n,k-1)+2$ (not lying necessarily in a curve of degree k, $k \leq n-1$):

Corollary 2. Let $\mathcal{X}$ be a n-independent point set with $\#\mathcal{X} \geq d(n,k-1)+2$ and $k \leq n-1$. Then there are at least $(N_k - 1)$ k-independent points in $\mathcal{X}$.

Proof. Note that what we need to prove is $H(k,\mathcal{X}) \geq N_k - 1$. First assume that there is a curve σ of degree k passing through all the nodes of $\mathcal{X}$ and, therefore, according to Theorem, we have

$$\dim \mathcal{P}_{k,\mathcal{X}} = 1.$$

Thus we obtain that

$$H(k,\mathcal{X}) = \dim \Pi_k - \dim \mathcal{P}_{k,X} = \dim \Pi_k - 1 = N_k - 1.$$

Now assume that there is no curve of degree k passing through all the nodes of $\mathcal{X}$. Then according to Lemma 2, we have

$$H(k,\mathcal{X}) \geq N_k. \qquad \square$$

In the next lemma we consider an arbitrary n-independent set $\mathcal{X}$ with $\#\mathcal{X} \leq d(n,k-1)+2$.

Corollary 3. Let $\mathcal{X}$ be a n-independent point set with $\#\mathcal{X} \leq d(n,k-1)+2$ and $k \leq n-1$. Then there are at least $\#\mathcal{X}-(n-k)(k-1)$ k-independent points in $\mathcal{X}$.

Proof. In view of Lemma 1, first let us enlarge the set $\mathcal{X}$ to an n-independent set $\bar{\mathcal{X}}$, $\#\bar{\mathcal{X}} = d(n,k-1)+2$. By Corollary 2, there is a subset $\mathcal{Y} \subset \bar{\mathcal{X}}$ of $(N_k - 1)$ k-independent points. Finally, let us remove from $\mathcal{Y}$ all the points belonging to the set $\bar{\mathcal{X}} \setminus \mathcal{X}$. Evidently, the resulted set is k-independent, and contains at least

$$(N_k - 1) - (\#\bar{\mathcal{X}} - \#\mathcal{X}) = \#\mathcal{X} - (n-k)(k-1)$$

points. □

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