PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2013, № 1, p. 6–12

Mathematics

ON n-INDEPENDENT SETS LOCATED ON QUARTICS

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Denote the space of all bivariate polynomials of total degree $\leq n$ by Π_n. We study the n-independence of points sets on quartics, i.e. on algebraic curves of degree 4. The n-independent sets $\mathcal{X}$ are characterized by the fact that the dimension of the space $\mathcal{P}_{\mathcal{X}} := \{p \in \Pi_n : p(x) = 0, \forall x \in \mathcal{X}\}$ equals $\dim \Pi_n - \#\mathcal{X}$. Next, polynomial interpolation of degree n is solvable only with these sets. Also the n-independent sets are exactly the subsets of Π_n-poised sets. In this paper we characterize all n-independent sets on quartics. We also characterize the set of points that are n-complete in quartics, i.e. the subsets $\mathcal{X}$ of quartic δ, having the property $p \in \Pi_n,\ p(x) = 0\ \forall x \in \mathcal{X} \Rightarrow p = \delta q,\ q \in \Pi_{n-4}$.

MSC2010: 41A10,41A63; 14H50.

Keywords: algebraic curve, fundamental polynomial, n-independent point set, n-complete point set.

1. Independent Point Sets. Denote by $\Pi_n = \Pi_n(\mathbb{R}^2)$ the space of bivariate algebraic polynomials of total degree not exceeding n:

$$\Pi_n = \left\{ p(x,y) = \sum_{i+j \leq n} a_{ij} x^i y^j,\ a_{ij} \in k \right\},$$

where $k = \mathbb{R}$ or $\mathbb{C}$. We have that

$$N := N_n := \dim \Pi_n = \binom{n+2}{2}.$$

Let us fix a set of points

$$\mathcal{X}_s := \{(x_1,y_1), \ldots, (x_s,y_s)\}.$$

The problem of finding $p \in \Pi_n$ satisfying the conditions

$$p(x_i,y_i) = c_i, \quad i = 1,2,...,s, \tag{1}$$

is called interpolation problem and denoted briefly by $(\Pi_n, \mathcal{X}_s)$. The polynomial p is called a data interpolating or just interpolating polynomial.

Definition 1.1. The interpolation problem $(\Pi_n, \mathcal{X}_s)$ is called solvable, if for any set of values $\{c_1, c_2, \ldots, c_s\}$ there exists a polynomial $p \in \Pi_n$ satisfying the conditions (1).

We call a polynomial $p \in \Pi_n$ fundamental or n-fundamental for the point $A = (x_k, y_k)$, and denote it by $p_k^* := p_A^* := p_{A,\mathcal{X}_s,n}^*$, if

$$p(x_i,y_i) = \delta_{ki}, \quad i = 1,2,...,s,$$

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where δ is the Kronecker symbol. Evidently a polynomial (from Π_n) vanishing at all points, but A is a constant times the fundamental polynomial. Sometimes it is convenient for us to call such polynomials fundamental too.

Definition 1.2. A set $\mathcal{X}$ is called Π_n-independent, or briefly n-independent, if all its fundamental polynomials $p_A^* \in \Pi_n$, $A \in \mathcal{X}$, exist.

Since the fundamental polynomials are linearly independent, we get that the following is a necessary condition for n-independence: $s \le N$. In the case of independence we have the following Lagrange formula for a polynomial satisfying interpolating conditions (1):

$$p = \sum_{i=1}^{s} c_i p_i^*.$$

Next, we bring the first characterization of n-independence:

Proposition 1.1. A set $\mathcal{X}$ is n-independent, if and only if the interpolation problem $(\Pi_n, \mathcal{X})$ is solvable.

Indeed, one side here follows from the Lagrange formula and another side follows from the fact that the fundamental polynomials are solutions of particular interpolation problems. □

Now let us discuss the poisedness.

Definition 1.3. A set $\mathcal{X}_s$ is called Π_n-poised, or briefly n-poised, if for any set of values $\{c_1, c_2, \dots, c_s\}$ there exists a unique polynomial $p \in \Pi_n$ satisfying the conditions (1). Evidently, a necessary condition for the poisedness $s = N$. Denote by $p|_{\mathcal{X}}$ the restriction of p on $\mathcal{X}$. By using an elementary linear algebra fact, we get

Proposition 1.2. The interpolation problem $(\Pi_n, \mathcal{X}_N)$ is not poised, if and only if

$$\exists p \in \Pi_n \quad p \ne 0, \quad p|_{\mathcal{X}} = 0.$$

Or, in other words, the interpolation problem $(\Pi_n, \mathcal{X}_N)$ is not poised, if and only if there is an algebraic curve of degree $\le n$ passing through all the points of $\mathcal{X}_N$.

The following proposition characterizes n-independent sets as subsets of n-poised sets (see, e.g., [1], Lemma 2.1).

Proposition 1.3. Any n-independent set $\mathcal{X}_s$ with $s < N$ can be enlarged to an n-poised set. Furthermore, a set $\mathcal{X}_N$ is n-poised, if and only if it is n-independent.

Now let us consider the following important polynomial class

$$\mathcal{P}_{n,\mathcal{X}} := \{p \in \Pi_n : p|_{\mathcal{X}} = 0\}.$$

The third characterization of n-independence is the following (see, e.g., [1], Eq. (2.3))

Proposition 1.4. A set $\mathcal{X}$ is n-independent, if and only if the following equality holds

$$\dim \mathcal{P}_{n,\mathcal{X}} = \dim \Pi_n - \#\mathcal{X}.$$

Next we bring the characterization of n-independence of $\mathcal{X}_s$ in terms of linear independence of some s vectors in k^N. Let associate any point (x, y) with its N-vector:

$$[x, y]_N := \left(1, x, y, \dots, x^n, x^{n-1}y, \cdots, y^n\right).$$

The *Vandermonde matrix* $V_n(\mathcal{X}_s)$ for the point set $\mathcal{X}_s$ and Π_n is an $s \times N$-matrix, whose rows are the N-vectors of the points of $\mathcal{X}_s$:

$$V_n(\mathcal{X}_s) = \begin{pmatrix} 1 & x_1 & y_1 & \cdots & x_1^n & x_1^{n-1}y_1 & \cdots & y_1^n \\ 1 & x_2 & y_2 & \cdots & x_2^n & x_2^{n-1}y_2 & \cdots & y_2^n \\ \vdots & & & & \vdots & & & \vdots \\ 1 & x_s & y_s & \cdots & x_s^n & x_s^{n-1}y_s & \cdots & y_s^n \end{pmatrix}.$$

We have the following (see, e.g., [1], Section 2.1)

Proposition 1.5. A set $\mathcal{X}$ is n-independent, if and only if the set of N-vectors associated with the points of $\mathcal{X}$ are linearly independent or, in other words, the *Vandermonde matrix* $V_n(\mathcal{X})$ has full row rank.

Let us define the Hilbert function of $\mathcal{X}$: $\mathcal{H}_n(\mathcal{X})$ as the cardinality of a maximal n-independent subset of $\mathcal{X}$, i.e.

$$\mathcal{H}_n(\mathcal{X}) := \max\{\#\mathcal{Y} : \mathcal{Y} \subset \mathcal{X},\ \mathcal{Y} \text{ is } n\text{-independent}\}.$$

We get readily from Proposition 1.4 that for any set of points $\mathcal{X}$

$$\dim \mathcal{P}_{n,\mathcal{X}} = \dim \Pi_n - \mathcal{H}_n(\mathcal{X}). \tag{2}$$

We use the same letter, say p, to denote a polynomial $p \in \Pi_n$ and the curve, for which $p(x,y) = 0$ is an equation. More precisely, suppose p is a polynomial of degree n without multiple factors. Then the curve of degree n defined by the equation $p(x,y) = 0$ we call also p.

Now let us consider n-completeness of a point set.

Definition 1.4. Let q be an algebraic curve of degree k without multiple components. A set $\mathcal{X}$ is called n-complete in q, if

$$p \in \Pi_n,\ p|_{\mathcal{X}} = 0 \Rightarrow p = qr,\ r \in \Pi_{n-k}. \tag{3}$$

Let us define for $k \leq n$

$$d(n,k) := \dim \Pi_n - \dim \Pi_{n-k} = \frac{1}{2} k(2n+3-k).$$

Let q be an algebraic curve of degree $k \leq n$. Consider the following polynomial class

$$\mathcal{P}_{n,q} := \{p \in \Pi_n : p = qr,\ r \in \Pi_{n-k}\}.$$

Obviously $\dim \Pi_{n,q} = \dim \Pi_{n-k}$.

It is well-known, that if q has no multiple component, then

$$p \in \Pi_n,\ p|_q = 0 \Rightarrow p = qr,\ r \in \Pi_{n-k}.$$

Notice that

$$\mathcal{P}_{n,q} \subset \mathcal{P}_{n,\mathcal{X}}\ , \text{ if } \mathcal{X} \subset q. \tag{4}$$

Therefore, the following holds for n-completeness of $\mathcal{X} \subset q$:

$$\mathcal{P}_{n,q} = \mathcal{P}_{n,\mathcal{X}}\ , \text{ if and only if } \dim \mathcal{P}_{n,q} = \dim \mathcal{P}_{n,\mathcal{X}}. \tag{5}$$

The latter condition, in view of (2), means that $\mathcal{H}_n(\mathcal{X}) = d(n,k)$. This implies (see, [2], Proposition 3.1.)

Proposition 1.6. Let q be an algebraic curve of degree $k \leq n$ without multiple components. Then the following hold.

i) If a subset $\mathcal{X}$ of q is n-independent, then $\#\mathcal{X} \leq d(n,k)$.

ii) If a subset $\mathcal{X}$ of q is n-complete, then $\#\mathcal{X} \geq d(n,k)$.

iii) A subset $\mathcal{X}$ of q containing $d(n,k)$ points is n-independent, if and only if it is n-complete in q.

iv) A subset of q containing more than nk points is n-complete in q, if q is irreducible.

Note that to prove iv) we use the fact that two curves of degrees n and k, without a common component, intersect at most at nk distinct points. Since any subset of an n-independent set is also n-independent, this proposition implies that in any n-poised set, at most $d(n,k)$ points can belong to a curve of degree k.

2. Some Results on Independent Sets. By conic, cubic and quartic we mean algebraic curves of degree $2,3$ and 4, respectively. Reducible conic is a pair of lines, while reducible cubic is a triple of lines or a pair of a line and an irreducible conic. We denote by $\alpha, \beta, \gamma, \delta$

lines, conics, cubics, quartics and the corresponding polynomials, respectively. The following is an evident continuity fact.

Lemma 2.1. Suppose that a set of points $\mathcal{X}_s$ is n-independent. Then there exists a positive number $\varepsilon > 0$ such that any set $\mathcal{X}'_s$ is n-independent, if the distance between x_i and x'_i is less than ε, $i = 1, \dots, s$.

In the next section we will use the following well-known result (see, e.g., [1], Proposition 4.1 and Theorem 4.4; [3], Theorems CB1-CB5).

Theorem 2.1. (Ceyley-Bacharach) Suppose that the curves p and q of degree m and n, respectively, intersect at exactly mn distinct points, i.e. $\#Z = mn$ with $Z = p \cap q$. Set $k_0 := n + m - 3$. Then the following hold.

i) Any polynomial from Π_{k_0} vanishing at $mn - 1$ intersection points vanishes also at the remaining intersection point.

ii) A subset $U \subset Z$, with $\#U = \mathcal{H}_k(Z)$, is k-independent, if and only if the set $U^c := Z \setminus U$ is $(k_0 - k)$-independent.

From now on, till the end of this section, we bring a relevant material from [4], which will be used in the sequel.

In the next lemma and proposition we have two sets $\mathcal{X}, \mathcal{Y}$, where $\mathcal{X}$ is n-independent and $\mathcal{Y}$ satisfies some conditions guaranteeing that the union set $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$ is n-independent.

Lemma 2.2. Suppose that a set of points $\mathcal{X}$ is n-independent and the points of another set $\mathcal{Y}$ have n-fundamental polynomials with respect to the set $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$. Then the set $\mathcal{Z}$ is n-independent.

Proposition 2.1. Suppose that a set $\mathcal{X} \subset q$ with $q \in \Pi_k$ is n-independent, $k \leq n$, and another set $\mathcal{Y}$ with $\mathcal{Y} \cap q = \emptyset$ is $(n-k)$-independent. Then the set $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$ is n-independent. The following proposition (see [3], Proposition 1; [5], Theorem 9) we will use frequently in the next sections.

Proposition 2.2. A set of $\leq 2n+1$ points on the plane is n-independent, if and only if no $n+2$ of them are collinear.

From here we readily get the following two corollaries:

Corollary 2.1. The following statements hold true.

i) Any set of $\leq n+1$ points is n-independent.

ii) Any set of $\geq n+2$ points located on a line is n-dependent.

iii) Any set of $n+1$ points located on a line is n-complete there.

Corollary 2.2. The following statements hold true for conics.

i) Any set of $\leq 2n+1$ points on an irreducible conic is n-independent.

ii) Any set of $\geq 2n+2$ points on a conic is n-dependent.

iii) Any set of $2n+1$ points located on an irreducible conic β is n-complete there.

Finally let us bring a result from [4] concerning the cubics.

Proposition 2.3. Let γ be any cubic. Then a set $\mathcal{X}$ of $\leq 3n$ points located on γ is n-dependent, if and only if one of the following holds:

i) a line component of γ contains $n+2$ points of $\mathcal{X}$, if γ is reducible;

ii) a conic component (possibly pair of lines) of γ contains $2n+2$ points of $\mathcal{X}$, if γ is reducible;

iii) $\#\mathcal{X} = 3n$ and there is a curve $\sigma \in \Pi_n$ such that $\gamma \cap \sigma = \mathcal{X}$.

3. Independence of Sets on Quartics. We characterize independence of sets of points located on a quartic in the following steps.

Step 1. Independence of Sets of $4n$–1 Points. Let us start by stating a special case of Proposition 1.6, i) when $k = 4$ and $d(n,4) = 4n - 2$.

• Any set of $\geq 4n-1$ points located on a quartic is n-dependent.

Step 2. Independence of Sets of $4n$–2 Points. By using Proposition 1.6, iii), we get a characterization for n-independence of sets of $4n-2$ points:

• A set of $4n-2$ points located on a quartic is n-independent, if and only if it is n-complete there.

Step 3. Independence of Sets of $4n$–3 Points. We characterize independence of $4n-3$ points located on a quartic in the following

Proposition 3.1. Let δ be any irreducible quartic. Assume that a set $\mathcal{X}$ of $4n-3$ points is located on δ and the points $A_i \in \delta \setminus \mathcal{X},\ i=1,2,3,$ are collinear. Then the set $\mathcal{X}$ is n-independent, if and only if a set $\mathcal{X}\cup\{A_i\}$ of $4n-2$ points is n-independent, where $i=1,2,3.$

Proof. Suppose that a set $\mathcal{X}\cup\{A_i\}$ is n-independent, where $1\le i\le 3$. Then, of course, $\mathcal{X}$ is n-independent. Now suppose by way of contradiction that the set $\mathcal{X}$ is n-independent, but each of the sets $\mathcal{X}\cup\{A_i\}, i=1,2,3,$ is n-dependent. This, in view of Lemma 2.2, means that

$$p\in\Pi_n,\ p|_{\mathcal{X}}=0 \Rightarrow p(A_i)=0,\ i=1,2,3.$$

Then, suppose that $q\in\Pi_n, q|_{\mathcal{X}}=0$. Let us prove that q and δ have a common component. Assume by way of contradiction that q and δ intersect only at finitely many points. Then we have that

$$q\cap\delta=\mathcal{X}\cup\{A_1,A_2,A_3\}.$$

Now, by using Theorem 2.1, ii), with the curves $q,\delta,\ k=1$ and $U=\{A_1,A_2,A_3\}$, we get that $U^c=\mathcal{X}$ is n-dependent ($k_0-1=n+4-3-1=n$), which is a contradiction.

Thus, q and δ have a common component. Next, since δ is irreducible, we get that q is divisible by δ. Therefore, we may conclude that $\mathcal{X}$ is n-complete in δ. This is a contradiction in view of Proposition 1.6, ii). □

Step 4. Independence of Sets of $4n$-4 Points. We characterize independence of any $4n-4$ points located on a quartic in the following two propositions.

Proposition 3.2. Let δ be a quartic. Suppose that a set $\mathcal{X}$ of $4n-4$ points located on δ is $(n-1)$-complete in δ :

$$p\in\Pi_{n-1},\ p|_{\mathcal{X}}=0\Rightarrow p=\delta q,\ q\in\Pi_{n-5}. \tag{6}$$

Suppose also that no line component of δ contains $n+2$ points of $\mathcal{X}$, if δ is reducible with a line component. Then $\mathcal{X}$ is n-independent.

Proof. Consider the following polynomial space:

$$\mathcal{P}_{n-5,\delta}=\{\delta q,\ q\in\Pi_{n-5}\}.$$

We get from (6) that $\mathcal{P}_{n-1,\mathcal{X}}\subset\mathcal{P}_{n-5,\delta}$. Therefore,

$$\dim\mathcal{P}_{n-1,\mathcal{X}}\le\dim\mathcal{P}_{n-5,\delta}=\dim\Pi_{n-5}.$$

Now, by using (2), we get

$$\dim\Pi_{n-1}-\mathcal{H}_{n-1}(\mathcal{X})\le\dim\Pi_{n-5}.$$

Therefore, we conclude that $\mathcal{H}_{n-1}(\mathcal{X})\ge 4n-6$.

Thus, we get that there is an $(n-1)$-independent subset $\mathcal{Y}$ of $\mathcal{X}$ of cardinality $4n-6$. Now consider the line α passing through the two points of $\mathcal{X}\backslash\mathcal{Y}$. According to the condition of the Proposition, α passes through at most $n+1$ points of $\mathcal{X}$. Now to complete the proof, we apply Proposition 2.1. Indeed, then we get that $\mathcal{X}$ is n-independent. □

Proposition 3.3. Let δ be a quartic. Suppose that a set $\mathcal{X}$ of $4n-4$ points located on δ is n-independent, and no cubic component of δ contains $3n$ points of $\mathcal{X}$, if δ is reducible with a cubic component. Then $\mathcal{X}$ is $(n-1)$-complete in δ.

Proof. Let us add two points A and B to $\mathcal{X}$ in δ, so that the resulted set $\mathcal{X}''$ of $4n-2$ points is n-independent. Assume that the line passing through A and B, which we denote by

α'', does not coincide with any line component of the quartic δ. Now suppose $p \in \Pi_{n-1}$ and $p|_{\mathcal{X}} = 0$. Consider the polynomial $q = \alpha'' p$. Then we have that q vanishes at all points of $\mathcal{X}''$. Therefore, according to Proposition 1.6, iii), we have that

$$q = \alpha'' p = \delta r, \text{ where } r \in \Pi_{n-4}.$$

Since α'' is not a component of δ, we conclude from here that it is a component of r, i.e. $r = \alpha'' s,\ s \in \Pi_{n-5}$. Thus, we conclude that $p = \delta s,$ where $r \in \Pi_{n-5}$ or, in other words, $\mathcal{X}$ is $(n-1)$-complete in δ.

Now to complete the proof it suffices to show that one can chose the two points such that the line α'' is not a component of the quartic δ. Suppose that the first point A is added to $\mathcal{X}$ in δ, so that the resulted set $\mathcal{X}'$ of $4n-3$ points is n-independent. If A belongs to a component, which is not a line, then we are done, since then the line α'' is not a component of δ.

Thus, suppose that A belongs to a line component α of δ. Let $\delta = \gamma\alpha$, where $\gamma \in \Pi_3$. We may assume, in view of Lemma 2.1, that $A \in \alpha \setminus \gamma$. Indeed, if A is an intersection point of α and γ, then by a small shift within γ we can remove A from α. It remains to show that one can chose the second point B from the cubic γ. Suppose it is not possible. Then, in view of Lemma 2.2, we have that

$$p \in \Pi_n,\ p|_{\mathcal{X}'} = 0 \Rightarrow p = \gamma r,\ r \in \Pi_{n-3}. \tag{7}$$

Now notice that there are $\geq n-3$ points of $\mathcal{X}$ on $\alpha \setminus \gamma$, since, otherwise, we would have $\geq 3n$ points of $\mathcal{X}$ on cubic component γ of δ. Also the point A was chosen from there. Hence, we get from (7) that r vanishes at these $n-2$ points, therefore, $r = \alpha s$. Thus, we conclude that

$$p \in \Pi_n,\ p|_{\mathcal{X}'} = 0 \Rightarrow p = \gamma\alpha s = \delta s,\ s \in \Pi_{n-4}. \tag{8}$$

This, in view of Proposition 1.6, is a contradiction. □

Finally let us turn to

Step 5. Independence of Sets of 4*n*-5 Points. We characterize independence of any $\leq 4n-5$ points located on an irreducible quartic in the following

Proposition 3.4. Let δ be any irreducible quartic. Then any set $\mathcal{X}$ of $\leq 4n-5$ points located on δ is n-independent.

Proof. Assume by way of contradiction that $\mathcal{X}$, with $\#\mathcal{X} = 4n-5$, is n-dependent. Then for any $A \in \delta \setminus \mathcal{X}$ we have that the set $\mathcal{X}' = \mathcal{X} \cup \{A\}$ is n-dependent, too. The set $\mathcal{X}'$ is not $(n-1)$-complete, i.e. there is a polynomial $q \in \Pi_{n-1}, q|_{\delta} \neq 0$, such that $q|_{\mathcal{X}'} = 0$, in view of Proposition 3.3. Thus, in view of the Bezout Theorem, we get

$$q \cap \delta = \mathcal{X}'.$$

Therefore, by using Theorem 2.1 (the Ceyley-Bacharach Theorem), we get

$$p \in \Pi_n, p|_{\mathcal{X}} = 0 \Rightarrow p|_{\mathcal{X}'} = 0 \quad (n = (n-1)+4-3).$$

Therefore, we have $p \in \Pi_n, p|_{\mathcal{X}} = 0 \Rightarrow p(A) = 0$. Since A was arbitrary point in $A \in \delta \setminus \mathcal{X}$ we conclude that $\mathcal{X}$ is n-complete in δ. This, in view of Proposition 1.6, is a contradiction. □

4. Completeness of Sets on Quartics. We will characterize the completeness of any set of points located on a quartic in the following steps.

Step 1. Completeness of Sets of 4*n*–3 Points. Let us start by stating a special case of Proposition 1.6, ii) when k=4. We have that $d(n,4) = 4n-2$. Therefore, we have the following:

- Any set of $\leq 4n-3$ points located on a quartic is not n-complete there.

Step 2. Completeness of Sets of 4*n*–2 Points. A characterization for n-completeness of sets of $4n-2$ points located on a quartic is stated in Step 2 of Section 3.

Step 3. Completeness of Sets of 4*n*–1 or 4*n* Points. We characterize completeness of sets of $4n-1$ and $4n$ points located on quartics in the following two propositions, respectively.

Proposition 4.1. Let δ be any quartic. Then any set $\mathcal{X}$ of $4n-1$ points located on δ is n-complete there, if and only if there is a point $A \in \mathcal{X}$ such that the set $\mathcal{X} \setminus A$ is complete there.

Proof. Suppose by way of contradiction that the set $\mathcal{X}$ of $4n-1$ points located on δ is n-complete there, while for any point $A \in \mathcal{X}$ the set $\mathcal{X} \setminus A$ is not complete. This means that for any $A \in \mathcal{X}$ there is a polynomial $p_A \in \Pi_n$ such that $p_A|_\delta \neq 0$ and $p_A|_{\mathcal{X}\setminus A} = 0$. Now notice that $p_A(A) \neq 0$, since otherwise the set $\mathcal{X}$ will be not complete in δ. Thus, we get that p_A is a fundamental polynomial for $A \in \mathcal{X}$. Since A is any point from $\mathcal{X}$, we get that $\mathcal{X}$ n-independent which, in view of Step 1 of Section 3, is a contradiction. □

Proposition 4.2. Let δ be any quartic. Then any set $\mathcal{X}$ of $4n$ points located on δ is n-complete there, if and only if there is a point $A \in \mathcal{X}$ such that the set $\mathcal{X} \setminus A$ is complete there.

The proof is identical with the previous one.

Finally let us turn to

Step 4. Completeness of Sets of $4n$+1 Points.

Proposition 1.6, iv) with k=4 gives a characterization for n-completeness of sets of $4n+1$ points located on a quartic. Namely, we have the following:

- Any set of $4n+1$ points located on an irreducible quartic is n-complete there.

The first author is supported by State Committee of Science of RA, grant # 11-1A290.

Received 20.12.2012

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