PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2012, № 1, p. 3–9

Mathematics

SOME PROPERTIES OF GENERALIZED EIGENFUNCTIONS OF ONE CLASS OF DIFFERENTIAL OPERATORS

H. A. ASATRYAN[1*], T. V. SARGSYAN[2**]

[1] *Chair of Differential Equations and Functional Analysis YSU, Armenia*
[2] *Institute of Mathematics of NAS Republic of Armenia*

In the paper considers continuality in a parameter of generalized eigenfunctions of one class of differential operators. Properties of one class of functions constructed by means of the generalized eigenfunctions and used in the theory of integral equations are studied.

Keywords: differential operator, generalized eigenfunction, spectral decomposition, integral equation.

1°. Let l be a $2n$-order $(n\in\mathbb{N})$ differential operation defined on the half-line $\mathbb{R}_+=(0,\infty)$ by the formula

$$\begin{aligned} l=&\frac{1}{i^{2n}}\cdot\frac{d^{2n}}{dx^{2n}}+\sum_{k=0}^{n-1}\frac{1}{i^{2k}}\cdot\frac{d^k}{dx^k}p_{2k}(x)\frac{d^k}{dx^k}+\\ &+\sum_{k=0}^{n-2}\frac{1}{2i^{2k+1}}\left\{\frac{d^k}{dx^k}p_{2k+1}(x)\frac{d^{k+1}}{dx^{k+1}}+\frac{d^{k+1}}{dx^{k+1}}p_{2k+1}(x)\frac{d^k}{dx^k}\right\}, \end{aligned} \tag{1}$$

where the coefficients p_k $(k=0,1,\dots,2n-2)$ are real functions satisfying

$$\int_0^1 x^{2n-1-k}\left|p_k(x)\right|dx+\int_1^\infty\left|p_k(x)-a_k\right|dx<\infty,\quad k=0,\dots,2n-2, \tag{2}$$

for some real constants a_k. When $n=1$ the second sum on the right side of (1) is assumed to be zero. For a function y defined on $\mathbb{R}_+$ an exact meaning of the expression $l(y)$ is defined by means of the quasi-derivatives $y^{[\nu]}$, $\nu=0,\dots,2n$ (see [1], as well as [2]). We will say that $l(y)$ makes sense, iff all the quasi-derivatives of y up to $(2n-1)$-th order exist and are absolutely continuous on each interval $[\alpha,\beta]\subset\mathbb{R}_+$. By $l(y)$ we will always mean $y^{[2n]}$, i.e. $l(y)=y^{[2n]}$.

Define a polynomial $Q(\xi)=\xi^{2n}+\sum_{k=0}^{2n-2}a_k\xi^k$, and denote by E the set of all complex numbers μ, for which the equation $Q(\xi)=\mu$ has complex multiple

[*] E-mail: asatryanh@yahoo.com
[**] E-mail: sargsyan.taron@gmail.com

roots. Let $\mu'_1 < \mu'_2 < \ldots < \mu'_{m'}$ be all real points of the set E. Assume that $\mu_1 < \mu_2 < \ldots < \mu_m$ are all real values of μ, for which the mentioned equation has real multiple roots. Let us also assume that $\mu_0' = \mu_0 = -\infty$, $\mu'_{m'+1} = \mu_{m+1} = \infty$. In each of the intervals (μ_k, μ_{k+1}) $(k = 0,1,\ldots,m)$ number of real roots of the equation $Q(\xi) = \mu$ is constant. Denote this number by $2r_k$. We note that $r_0 = 0$, $r_m = 1$. Further, we fix $k = 1,\ldots,m$ and for $\mu \in (\mu_k, \mu_{k+1}) \setminus E$ denote by $\xi_\nu(\mu)$ $(\nu = 1,\ldots,2n)$ the roots of this equation. Choosing $\varepsilon_0 > 0$, so that the strip $\{\mu; -\varepsilon_0 \le \operatorname{Im}\mu \le \varepsilon_0\}$ does not contain any non-real points of the set E, we assume that: for $\mu \in (\mu'_s, \mu'_{s+1}) \subset (\mu_k, \mu_{k+1})$ the functions ξ_ν $(\nu = 1,\ldots,2n)$ admit analytical continuation to the domain $\{\mu; -\varepsilon_0 < \operatorname{Im}\mu < \varepsilon_0,\ \mu'_s < \operatorname{Re}\mu < \mu'_{s+1}\}$; $\operatorname{Im}\xi_\nu(\mu) > 0$ $(\nu = 1,\ldots,2n)$ and $\operatorname{Im}\xi_\nu(\mu) < 0$ $(\nu = n+1,\ldots,2n)$ as soon as $0 < \operatorname{Im}\mu < \varepsilon_0$; for $\nu = 1,\ldots,r_k, n+1,\ldots,n+r_k$ the roots ξ_ν are real and admit analytical continuation to the domain $\{\mu; -\varepsilon_0 < \operatorname{Im}\mu < \varepsilon_0,\ \mu_k < \operatorname{Re}\mu < \mu_{k+1}\}$. Due to the construction, the functions ξ_ν increase in the interval (μ_k, μ_{k+1}) for $1 \le \nu \le r_k$ and decrease in the same interval for $n+1 \le \nu \le n+r_k$.

Suppose that $\mathcal{L}'$ and $\mathcal{L}_0$ are, respectively, the maximal and minimal closed symmetric operators generated by differential operation l, while D' and D_0 are their domains of definition.

We use the following notation: $[y,z]_x = \sum\limits_{s=0}^{2n-1} y^{[2n-1-s]}(x)\overline{z^{[s]}(x)}$ for functions y and z such that $l(y)$ and $l(z)$ are defined. For any $y, z \in D'$ the finite limit $[y,z]_0 = \lim\limits_{x\to 0}[y,z]_x$ exists. The definition domain of arbitrary self-adjoint extensions $\mathcal{L}$ of symmetric operator $\mathcal{L}_0$ consists of those and only those functions that satisfy the boundary conditions

$$[y, z_j]_0 = 0, \quad j = 1,2,\ldots,n, \tag{3}$$

where z_j, $j = 1,\ldots,n$, are some linearly independent by module D_0 functions of D' such that $[z_s, z_j]_0 = 0$, $s, j = 1,\ldots,n$.

The point spectrum T of the operator $\mathcal{L}$ is bounded from below and has no finite accumulation points other than μ_j, $j = 1,2,\ldots,m$ (see [2, 3]). The equation $l(y) = \mu y$ has exactly r_k linearly independent bounded solutions satisfying the boundary conditions (3) for $\mu \in (\mu_k, \mu_{k+1}) \setminus T$. Such solutions are uniquely determined up to a special asymptotics at infinity. Below we will consider a system of linearly independent solutions $\varphi_{1s}, \varphi_{2s}$ $(s = 1,2,\ldots,r_k)$ uniquely defined by the following asymptotics when $x \to \infty$:

$$\varphi_{1s}(x,\mu) = \frac{1}{\sqrt{2\pi}}\left\{\sqrt{\xi'_s(\mu)}e^{ix\xi_s(\mu)} + \sum_{j=n+1}^{n+r_k}\sqrt{-\xi'_j(\mu)}b_{sj}(\mu)e^{ix\xi_j(\mu)}\right\} + o(1), \tag{4}$$

$$\varphi_{2s}(x,\mu) = \frac{1}{\sqrt{2\pi}}\left\{\sum_{j=1}^{r_k}\sqrt{\xi'_j(\mu)}b_{sj}(\mu)e^{ix\xi_j(\mu)} + \sqrt{-\xi'_s(\mu)}e^{ix\xi_s(\mu)}\right\} + o(1). \tag{5}$$

We note that unitary matrices $B_1=(b_{s,n+j})_{s,j=1}^{r_k}$, $B_2=(b_{s,j})_{s,j=1}^{r_k}$ are uniquely defined (see [2, 3]). Solutions φ_{is} $(i=1,2;\ s=1,\dots,r_k)$ are called normalized generalized eigenfunctions of the operator $\mathcal{L}$, corresponding to the continuous spectrum. It is known (see [2, 3]), that these functions play an important role in the spectral decomposition of the operator $\mathcal{L}$ and, in particular, in formulas of the inversion by means of a minimum number of solutions of the equation $l(y)=\mu y$. These functions play an important role in the theory of solvability of integral equtions of $\mathcal{L}$-convolution (see [4–6]) as well. The present paper studies the problem of a continuous dependence on the parameter μ of the functions φ_{is} $(i=1,2;\ s=1,2,\dots,r_k)$, as well as properties of the functions of a form

$$K(x,t)=\int_{\mu_k}^{\mu_{k+1}} c(\mu)\varphi_{is}(x,\mu)\overline{\varphi_{is}(t,\mu)}d\mu,\qquad x>0,\ \ t>0,\tag{6}$$

playing important role in the abovementioned theory of integral operators of $\mathcal{L}$-convolution.

2°. The following theorem is true:

Theorem 1. For $\mu\in(\mu_k,\mu_{k+1})\setminus(E\cup T)$ and $x>0$ the functions $\varphi_{1s}(x,\mu)$, $\varphi_{2s}(x,\mu)$ $(s=1,\dots r_k)$ and their quasi-derivatives up to the order $2n-1$ are continuous.

If, in addition, for some $\nu_0\in\mathbb{N}$ the following conditions are fulfilled:

$$\int_0^\infty (1+t^{\nu_0})\left|p_k(t)-a_k\right|dt<\infty,\quad k=0,1,\dots,2n-2,\tag{7}$$

then the functions $\varphi_{1s}(x,\mu)$, $\varphi_{2s}(x,\mu)$ $(s=1,2,\dots,r_k)$ have continuous derivatives in μ up to the order ν_0 for $\mu\in(\mu_k,\mu_{k+1})\setminus(E\cup T)$.

Proof. We use the following notation:

$$[e^{ix\lambda}]^{[\nu]^0}=\lambda^\nu e^{ix\lambda}\quad(\nu=0,\dots,n),\ \ [e^{ix\lambda}]^{[2n]^0}=Q(\lambda)e^{ix\lambda},$$

$$[e^{ix\lambda}]^{[2n-1-\nu]^0}=\left(\lambda^{2n-1-\nu}+\frac{1}{2}a_{2\nu+1}\lambda^\nu+\sum_{k=2\nu+2}^{2n}a_k\lambda^{k-\nu-1}\right)e^{ix\lambda},\quad \nu=0,1,\dots,n-2.$$

As it is known (see [2, 3]), the equation $l(y)=\mu y$ has a fundamental system of solutions $y_j(x,\mu)$ $(j=1,\dots,2n)$, whose quasi-derivatives have the following asymptotics as $x\to\infty$:

$$y_j^{[\nu]}(x,\mu)=(e^{ix\xi_j(\mu)})^{[\nu]^0}+o(e^{ix\xi_j(\mu)}),\qquad \nu=0,\dots,m-1.\tag{8}$$

Define functions q_s, A_s, sets $M_j^{\pm}(\mu)$ and numbers $\delta_{-1},\delta_0,\delta_1$ by the following formulas:

$$q_{2k}=p_{2k}-a_{2k},\ q_{2k+1}=\frac{1}{2}(p_{2k+1}-a_{2k+1})\ \ (k=0,1,\dots,n-1),\ q_{-1}=0;$$

$$A_s(\mu)=\frac{i}{Q'(\xi_s(\mu))},\qquad s=1,2,\dots,2n;$$

$$M_j^+(\mu)=\{s:\ \mathrm{Im}(\xi_j(\mu)-\xi_s(\mu)\geq 0)\};$$

$$M_j^-(\mu)=\{s:\ \mathrm{Im}(\xi_j(\mu)-\xi_s(\mu)<0)\};$$

$$M_j^+(\mu)\cup M_j^-(\mu)=\{1,2,...,2n\}\ ;\ \delta_{-1}=1\ ,\ \delta_0=\delta_1=0\ .$$

Quasi-derivatives $y_j^{[\nu]}$ ($\nu=0,1,...,2n-1$) can be restored from the equalities

$$\begin{aligned}y_j^{[\nu]}(x,\mu)=(e^{ix\xi_j})^{[\nu]^0}+\sum_{\tau=-1}^{1}\sum_{r=\delta_\tau}^{n-1+\delta_\tau}\sum_{s\in M_j^+}A_s\xi_s^{r+\tau}(e^{ix\xi_s})^{[\nu]^0}\int_x^\infty e^{-it\xi_s}q_{2r+\tau}(t)y_j^{[r]}(t,\mu)dt-\\ -\sum_{\tau=-1}^{1}\sum_{r=\delta_\tau}^{n-1+\delta_\tau}\sum_{s\in M_j^-}A_s\xi_s^{r+\tau}(e^{ix\xi_s})^{[\nu]^0}\int_\alpha^x e^{-it\xi_s}q_{2r+\tau}(t)y_j^{r}(t,\mu)dt,\end{aligned}\tag{9}$$

where $\alpha>0$.

It is easy to obtain from here systems of integral equations for the functions $z_{j,\nu}=e^{-ix\xi_j}y_j^{[\nu]}$ when $j=1,2,...,2n$ and $\nu=0,1,...,n$. Taking into account that for $j=1,...,r_k,n+1,...,n+r_k$ the sets $M_j^\pm(\mu)$ in (9) and in the mentioned systems do not change on the intervals $[\mu',\mu'']\subset(\mu_k,\mu_{k+1})\setminus E$ as functions of μ, it is not difficult to see that for x large enough the successive approximations method is applicable to these systems, that leads to the continuous solution in $\mu\in[\mu',\mu'']$. Consequently, the functions $y_j^{[\nu]}$ ($\nu=0,...,n$) are also continuous for x large enough and $\mu\in[\mu',\mu'']$. The formula (9) implies the continuity of the functions $y_j^{[\nu]}$ on this set when $\nu=n+1,...,2n$ as well. By similar arguments it is easily seen that when the conditions (7) are satisfied, the functions $y_j^{[\nu]}$ are continuously differentiable with respect to $\mu\in[\mu',\mu'']$ up to the order ν_0. By virtue of Lemma 1.1.4 [2], the proved assertions remain valid for all $x>0$. It is known (see Lemma 1.1.5 [2]), that the equation $l(y)=\mu y$ possesses linearly independent solutions $h_j(x,\mu)$ $(j=r_k+1,...,n)$, whose quasi-derivatives are analytical in some neighborhood of interval $[\mu',\mu'']$ as soon as $[\mu',\mu'']\subset(\mu_k,\mu_{k+1})\setminus E$ and have the asymptotics $h_j^{[\nu]}(x,\mu)=o(1)e^{-bx}$, $x\to\infty$, where $b=\min\limits_{\mu\in[\mu',\mu'']}\min\limits_{r_k+1\le s\le n}\{\operatorname{Im}\xi_s(\mu)\}>0.$ Since $h_j\in\operatorname{span}\{y_{r_k+1},...,y_n\}$, then, redenoting y_j $(j=1,...,r_k,n+1,...,n+r_k)$ by h_j, we obtain that the set of bounded solutions of the equation $l(y)=\mu y$ coincides with $\operatorname{span}\{h_j;j=1,...,n+r_k\}$. Therefore, there exist functions $c_{sj}(\mu)$ $(s,j=1,...,n+r_k)$ such that $\varphi_{2s}(x,\mu)=\sum\limits_{j=1}^{n+r_k}c_{sj}(\mu)h_j(x,\mu)$. Requirements $[\varphi_{2s},z_j]_0=0$ $(j=1,...,n)$ along with formula (5) for each $s=1,...,r_k$ lead to the following system of linear algebraic equations with respect to coefficients c_{sj}:

$$\sum_{j=1}^{n}c_{sj}(\mu)[h_j,z_r]_0=-\sqrt{-\xi_{s+n}(\mu)}[h_{s+n},z_r]_0,\quad r=0,1,...,n.$$

It is easy to see that the determinant of the matrix $([h_j,z_r]_0)_{j,r=1}^n$ is not zero as soon as $\mu\in T$. Hence, the statement for functions φ_{2s} of the Theorem follows.

Formulas (4), (5) imply $c_{sj}=\dfrac{1}{\sqrt{2\pi}}\sqrt{\xi'_j}\,b_{s,j}$ $\quad(s,j=1,\dots,r_k)$ and $\varphi_{2s}=\sum\limits_{j=1}^{r_k}b_{s,j}\varphi_{1s}$ $(s=1,\dots,r_k)$. Taking into account nonsingularity of the matrix B_2, we obtain the assertion of the Theorem also for functions φ_{1s}. The proof is completed.

3$^{\mathrm{o}}$. Next, we give some properties of the functions of the form (6), assuming i and s are fixed $(i=1,2;\ s=1,\dots,r_k)$.

Theorem 2. Suppose $c(\mu)$ is continuous on the interval $[\mu_k,\mu_{k+1}]$ and vanishes in some neighborhood of the set $(E\cup T)\cap[\mu_k,\mu_{k+1}]$. Then the function K defined by the formula (6) has the following properties:

a) operation l is applicable to $K(x,t)$ in each variable and $l_x(K(x,t))=l_t^{\#}(K(x,t))$, where $l^{\#}$ is the differential operation defined by formula $l^{\#}(y)=\overline{l(\overline{y})}$, while an index at l and $l^{\#}$ indicates the variable, with respect to which the corresponding differential operation acts;

b) for each $x>0$ $\overline{K(x,t)}$, as a function of t, belongs to the domain of definition of the operator $\mathcal{L}$.

If, in addition, the conditions (8) are satisfied with $\nu_0=2$, $c(\mu)$ is a twice continuously differentiable function, and the functions

$$\int\limits_x^{\infty}|p_k(t)-a_k|dt,\quad k=0,1,\dots,2n-2,$$

belong to $L^1(\mathbb{R}_+)$, then the following assertions are true as well:

c) quasi-derivatives $K_x^{[\nu]}(x,t)$ in the variable x $(\nu=0,\dots,2n)$ are continuous in both variables, while integrals

$$\int\limits_0^{\infty}\left|K_x^{[\nu]}(x,t)\right|dt,\qquad \nu=0,\dots,2n, \tag{10}$$

converge uniformly in x on each interval $[\alpha,\beta]\subset\mathbb{R}_+$;

d) the following relations are fulfilled:

$$\lim_{x\to 0}\int\limits_0^{\infty}\left|[K(x,t),z_j(x)]_x\right|dt=0,\qquad j=1,2,\dots,n, \tag{11}$$

where z_j are the functions appearing in (3);

e) for any $\beta>0$

$$\lim_{x\to\infty}\int\limits_0^{\beta}|K(x,t)|dt=0. \tag{12}$$

Proof. Properties a) and b) can be verified directly.

We denote $\beta_j=\min\limits_{s\in M_j^-}\ \min\limits_{\mu\in[\mu',\mu'']}\{\mathrm{Im}(\xi_s(\mu)-\xi_j(\mu))\}$ $(j=1,\dots,2n)$. It is obvious that $\beta_j>0$, and for $j=1,\dots,r_k,n+1,\dots,n+r_k$, $r\le n$, $s\in M_j^-$ we have

$$\int\limits_{\alpha}^{x/2}e^{i(x-t)\xi_s(\mu)}q_{2r+\tau}(t)y_j^{[r]}(t)dt\le Me^{-x\beta j/2}\int\limits_{\alpha}^{x}|q_{2r+\tau}(t)|dt$$

for some constant M. Hence, by virtue of (8), (9), one can see that the following representation is valid:

$$\varphi_{is}^{[\nu]}(x,\mu)=\left(\sum_{j=1}^{r_k}+\sum_{j=n+1}^{n+r_k}\right)g_{\nu sj}(\mu)e^{ix\xi_j(\mu)}+G_{\nu s}(x,\mu),\quad \nu=0,\dots,2n-1, \tag{13}$$

where $g_{\nu sj}(\mu)$, $G_{\nu s}(x,\mu)$ are twice continuously differentiable with respect to μ functions and the function $G(x,\mu)$ belongs to $L^1(\mathbb{R}_+)$ for each fixed μ. Besides, $|G(x,\mu)|\le \text{const } G_0(x)$, where $G_0\in L^1(\mathbb{R}_+)$ and $\lim\limits_{x\to\infty}G_0(x)=0$.

Using (13) and denoting $\lambda_{k,j}=\xi_j(\mu_k)$, $\lambda_{k+1,j}=\xi_j(\mu_{k+1})$, we obtain

$$K_x^{[\nu]}(x,t)=\int_{\mu_k}^{\mu_{k+1}}c(\mu)\varphi_{is}^{[\nu]}(x,\mu)\overline{G_0(t,\mu)}d\mu+$$

$$+\left(\sum_{j=1}^{r_k}+\sum_{j=n+1}^{n+r_k}\right)\int_{\lambda_{k,j}}^{\lambda_{k+1,j}}c(Q(\lambda))\varphi_{is}^{[\nu]}(x,Q(\lambda))g_{\nu sj}(Q(\lambda))Q'(\lambda)e^{-it\lambda}d\lambda .$$

Obviously, the functions $K_x^{[\nu]}$ are continuous in both variables. The summands in the first two sums for each $x>0$ are the Fourier transform of twice continuously differentiable finite functions and, therefore, as functions of t, belong to $L^1(\mathbb{R}_+)$. The last quantity is estimated in absolute value by means of function $G_0(t,\mu)$ and, therefore, as a function of t, also belongs to $L^1(\mathbb{R}_+)$. Consequently, the integrals $I_\nu(x,\alpha)=\int_\alpha^\infty\left|K_x^{[\nu]}(x,t)\right|dt$ $(\nu=0,\dots,2n)$ are continuous functions in the variables x,α $(x>0,\ \alpha>0)$ and monotonically decreasing in α with $\lim\limits_{\alpha\to\infty}I_\nu(x,\alpha)=0$ for each $x>0$. This immediately implies the uniform convergence of the integrals (11), which proves *c*).

It is easy to see that

$$A_j(x,t)=[K(x,t)z_j(x)]_x=\int_{\mu_k}^{\mu_{k+1}}c(\mu)[\varphi_{is}(x,\mu)z_j(x)]_x\overline{\varphi_{is}(t,\mu)}d\mu.$$

Since the integrand is continuous with respect to μ, t, α $(t>0,\ \alpha>0)$ and $\lim\limits_{x\to\infty}[\varphi_{is}(x,\mu)z_j(x)]_x=0$, then the function $A_j(x,t)$ $(j=1,\dots,n)$ is continuous for $x\ge 0$ and $t\ge 0$ and $A(0,t)\equiv 0$. Similarly to the previous paragraph one can prove that $A(x,t)$ belongs to $L^1(\mathbb{R}_+)$ with respect to the variable t for each fixed $x\ge 0$, while the integrals $J_\nu(x)=\int_0^\infty|A_\nu(x,t)|dt$ $(\nu=1,2,\dots,n)$ converge uniformly when $x\in[0,1]$. Thus, $J_\nu(x)$ is a continuous function and $\lim\limits_{x\to 0}J_\nu(x)=0$.

This proves the assertion d).

Using (13) we easily obtain the following representation:

$$K_i(x,t)=\int_{\mu_k}^{\mu_{k+1}} G(x,\mu)c(\mu)\overline{\varphi_{is}(t,\mu)}d\mu+$$
$$+\left(\sum_{j=1}^{r_k}+\sum_{j=n+1}^{n+r_k}\right)\int_{\lambda_{k,j}}^{\lambda_{k+1,j}} e^{ix\lambda}c(Q(\lambda))\overline{\varphi_{is}(t,Q(\lambda))}g_{0sj}(Q(\lambda))Q'(\lambda)d\lambda. \tag{14}$$

Obviously, $G_0(x)\to 0$ as $x\to\infty$ implies

$$\lim_{x\to\infty}\int_0^\beta\left|\int_{\mu_k}^{\mu_{k+1}} G(x,\mu)c(\mu)\overline{\varphi_{is}(t,\mu)d\mu}\right|dt=0 .$$

Each of the terms in the last two sums of (14) has a form $\int_{\lambda'}^{\lambda''} e^{ix\lambda}Y(t,\lambda)d\lambda$, where $Y(t,\lambda)$ is continuous in the rectangle $[0,\beta]\times[\lambda',\lambda'']$. By fixing $\varepsilon>0$ and choosing partition $\lambda'=\lambda_0'<\lambda_1'<\ldots<\lambda_n'=\lambda''$, so that $|Y(t,\lambda)-Y(t,\lambda')|<\varepsilon/2(\lambda''-\lambda')$ $(0\le t\le\beta,\ \lambda_{j-1}'\le\lambda\le\lambda_j',\ j=1,2,\ldots,n)$, following [7] (see Chap. 3, § 5) it is easy to see that $\int_0^\beta\left|\int_{\lambda'}^{\lambda''} e^{ix\lambda}Y(t,\lambda)d\lambda\right|dt\le\left(\frac{2M_0 n}{x}+\frac{\varepsilon}{2}\right)\beta$, where $M_0=\max|Y(t,\lambda)|$. Hence, the integral on the right side of the last inequality tends to zero.

Thus, the assertion e) is proved.

Received 05.12.2011

REFERENCES

1. **Naimark M.A.** Linear Differential Operators. M.: Nauka, 1969 (in Russian).
2. **Babasyan S.V.** The Inverse Scattering Problem for Differential Operators of Higher Orders. Phd. Thesis. Yer., 1992 (in Russian).
3. **Babasyan S.V., Khachatryan I.G.** // Dokladi AN Arm. SSR., 1988, v. 87, № 3, p. 111–114 (in Russian).
4. **Kamalyan A.G., Nersesyan A.B., Khachatryan I.G.** // Dokladi NAN Armenii, 2011, v. 111, № 1, p. 15–22.
5. **Kamalyan A.G., Khachatryan I.G., Nersesyan A.B.** // Izv. NAN Armenii. Matematika. Journal of Contemporary Mathematical Analysis, 1994, v. 29, № 6, p. 31–81.
6. **Kamalyan A.G., Nersesyan A.B., Khachatryan I.G.** In the book: The Third Russian-Armenian Workshop by Math. Physics, Compl. Analysis and Related Topics. Yer., 2010, p. 74–79.
7. **Olver F.** Introduction to Asymptotics and Special Functions, Moscow: Nauka, 1978 (in Russian).