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Mathematics

AN ASYMPTOTIC ESTIMATE OF THE NUMBER OF SOLUTIONS OF A SPECIAL SYSTEM OF BOOLEAN EQUATIONS

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In this paper a special class of systems of Boolean equations is investigated. For a "typical" case of such systems an asymptotic estimate for the number of solutions is determined.

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Introduction. Many problems of discrete mathematics, including the problems which are traditionally considered to be complex, lead to the solutions of the systems of Boolean equations of the form

$$\begin{cases} f_i(x_1,\dots,x_n)=0, \\ i=1,...,l, \end{cases} \tag{1}$$

or to the revealing of those conditions, under which the system (1) has a solution. In general, the problem of realizing whether the system (l) has a solution or not is NP-complete [1]. Therefore, it is often necessary to explore a number of solutions in the "typical" case, or to consider special classes of systems of equations, using their specificity.

Some Necessary Definitions. Let $\{M(n)\}_{n=1}^{\infty}$ be the collection of sets such that $|M(n)|_{\overrightarrow{n\to\infty}}\infty$ ($|M|$ is the power of the set M), and $M^s(n)$ be the subset of all elements from $M(n)$, which have a property S. We say that almost all elements of the set $M(n)$ have a property S, if $\left|M^S(n)\right|/|M(n)|_{\overrightarrow{n\to\infty}}\infty$.

Everywhere below the notation log stands for the logarithm of the base 2; $[a]$ denotes the integer part of a.

We denote by $S_{n,l}$ the set of all the systems of the form (1), where $f_i(x_1,\dots,x_n)$, $i=1,\dots,l$, are pairwise different Boolean functions in variables $x_1,x_2,\dots,x_n$. It is easy to see that $|S_{n,l}|=C_{2^{2^n}}^{l}$.

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Let $B=\{0,1\}$, $B^n=\{\tilde{\alpha}/\tilde{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_n),\alpha_i\in B,1\le i\le n\}$. The vector $\tilde{\alpha}_i=(\alpha_1,\alpha_2,\ldots,\ \alpha_n)\in B^n$ is called a solution of (1), if

$$\begin{cases} f_i(\alpha_1,\alpha_2,\ldots,\alpha_n)=0,\\ i=1,\ldots,l. \end{cases}$$

We denote by $t(S)$ the number of solutions of the system S. In [2,3] it is found the asymptotics of the number of solutions $t(S)$ for almost all the systems S of the set $S_{n,l}$ for whole range values of parameter l, as $n\to\infty$. Moreover, in [3] a general case, the systems of equations in k-valued logic was considered. In this work a class of systems of equations of the special form is considered. The asymptotic behavior of the number of solutions of equations systems in a "typical" case is found.

1. Let $L(n)$ denote the set of all linear Boolean functions, depending on the variables $x_1,x_2,\ldots,x_n$, i.e. $L(n)=\{c_0+c_1x_1+\cdots+c_nx_n\,/\,c_i=0,1;\ i=1,\ldots,n\}$, where $+$ indicates addition modulo.

2. Obviously, $|L(n)|=2^{n+1}$. Let $f_i=f_i(x_1,\ldots,x_n)=L_{i1}\ldots L_{im},\ i=1,\ldots,l$, where $L_{ij}=L_{ij}(x_1,\ldots,x_n)\in L(n),\ j=1,\ldots,m$.

Then the system of equations (1) takes the form

$$\begin{cases} L_{i1}\ldots L_{im}=0,\\ i=1,\ldots,l. \end{cases} \tag{2}$$

Let $R(n,l,m)$ denote the set of all systems of l equations of the form (2) ($f_i\neq f_j$, if $i\neq j$) and let $c(n,m)=C^m_{2^{n+1}}$. It is easy to see that $|R(n,l,m)|=C^l_{c(n,m)}$. For the numbers of the solutions $t(S)$of almost all systems S of $R(n,l,m)$ the following statement is true (here and further $f(n)\sim g(n)$ stands for the relation $f(n)/g(n)\xrightarrow[n\to\infty]{}1$, $f(n)=o(g(n))$ for $f(n)/g(n)\xrightarrow[n\to\infty]{}0$).

Theorem.

1. If $n-\ell(m-\log(2^m-1))\xrightarrow[n\to\infty]{}\infty$ and $l^2=o(c(n,m))$, $m^2=o\ (2^n)$, then for almost all systems S of $R(n,l,m)$ we have $t(S)\sim 2^n(1-2^{-m})^l$.

2. If $n-\ell(m-\log(2^m-1))\xrightarrow[n\to\infty]{}-\infty$ and $m^2=o\ (2^n)$, then almost all systems S of $R(n,l,m)$ have no solutions.

3. If $n-\ell(m-\log(2^m-1))$ is bounded as $n\to\infty$ and $l^2=o\ (c(n,m))$, $m^2=o\ (2^n)$, then for almost all systems of $R(n,l,m)$ the number of solutions $t(S)$ is bounded from above by an arbitrary function $\varphi(n)$, satisfying the condition $\varphi(n)\xrightarrow[n\to\infty]{}\infty$.

Proof. The following known or easily checking inequalities hold:

1. The first Chebyshev inequality [4]. Let the random variable ξ take the non-negative values and have mathematical expectation $M\xi$. Then for any $t>0$ we have

$$P(\xi\ge t)\le M\xi/t.$$

2. The second Chebyshev inequality [4]. Let the above random variable ξ have a dispersion $D\xi$. Then for any $t>0$ we have

$$P(|\xi-M\xi|\ge t)\le D\xi/t^2.$$

3. For any $x>1$

$$(1-1/x)^x<e^{-1}.$$

4. For any $x \geq 1$

$$(1-1/x)^x > e^{-1-\frac{1}{x}}.$$

5. For any natural n with $1 \leq m \leq n$ we have

$$C_n^m < \left(\frac{en}{m}\right)^m.$$

6. Let $b(k;n,p) = C_n^k p^k q^{n-k}$, where $0 < p,\ q < 1,\ p+q = 1$. Then for $r > np$

$$\sum_{j=0}^{n-r} b(r+j;n,p) < b(r;n,p)(r+1)q/(r+1-(n+1)p)$$

(the estimate of the “tail” of the binomial distribution [4]).
7. Bernoulli inequality: for any $x \geq -1$ and natural n, we have

$$(1+x)^n \geq 1+nx.$$

Consider another set $R'(n,l,m)$ of the systems of l equations of the form (2), where $L_{ij} = L_{ij}(x_1,\ldots,x_n) \in L(n)$, $i = 1,\ldots,l$, $j = 1,\ldots,m$ (a condition $f_i \neq f_j$ when $i \neq j$ is canceled). In addition, the systems of equations are assumed to be “ordered”, i.e. two systems S' and S'' from $R'(n,l,m)$ are different, if one of them is obtained from other by a transposition of non-equivalent equations. It's easy to observe that $|R'(n,l,m)| = c(n,m)^l$.

Let S be a system in $R(n,l,m)$. Arranging (transpositions by all the variations) the equations in S, we obtain $l!$ new systems, differing from each other by the transposition of the equations. Thus, from the set $R(n,l,m)$ we obtain a new set $R''(n,l,m)$ of ordered and non-repetitive (containing no pair of equivalent equations) systems. It is evident that $|R''(n,l,m)| = |R(n,l,m)|l!$. Using Bernoulli inequality, one can estimate the relation

$$\frac{|R''(n,l,m)|}{|R'(n,l,m)|} = \frac{l!\cdot|R(n,l,m)|}{(c(n,m))^l} =$$

$$= \frac{l!\cdot C^l_{c(n,m)}}{(c(n,m))^l} = \frac{c(n,m)\cdot(c(n,m)-1)\times\cdots\times(c(n,m)-l+1)}{(c(n,m))^l} =$$

$$= \prod_{i=0}^{l-1}\left(1-\frac{i}{c(n,m)}\right) \geq \left(1-\frac{l-1}{c(n,m)}\right)^l \geq 1-\frac{(l-1)}{c(n,m)}l \to 1,$$

if $l^2/c(n,m)\underset{n\to 0}{\longrightarrow}\infty$. Thus, if $l^2 = o(c(n,m))$, then any statement concerning the almost all systems of the set $R'(n,l,m)$ is also true for almost all the systems of the set $R''(n,l,m)$.

Let almost all systems of $R''(n,l,m)$ have property E, which is invariant with respect to the transpositions of the equations of system. It is easy to see that almost all the systems of the set $R(n,l,m)$ will also have property E. Thus, for the Proof of the Theorem when $l^2 = o\ (c(n,m))$ it will be enough to consider the set $R'(n,l,m)$ instead of $R(n,l,m)$.

Next, let S be a system of $R'(n,l,m)$. Arranging (transpositions by all the variations) linear functions in the equations of the system, we obtain $(m!)^l$ new ordered systems, different from each other by permutation of linear functions in the equations. Thus, from the set of $R'(n,l,m)$ a new set of $R'''(n,l,m)$ ordered systems can be obtained. It is clear that $|R'''(n,l,m)(n,l,m)| = (m!)^l \cdot |R'(n,l,m)|$, i.e.

$$\left|R'''(n,l,m)(n,k,p)\right| = \left(2^{n+1} \cdot \left(2^{n+1}-1\right) \times \ldots \times \left(2^{n+1}-m+1\right)\right)^l . \tag{3}$$

Finally, we denote by $R''''(n,l,m)$ the expansion of the set $R'''(n,l,m)$ (different systems of $R''''(n,l,m)$ are allowed to have same equations). It is easy to see that

$$\left|R''''(n,l,m)\right| = \left(2^{n+1}\right)^{lm} . \tag{4}$$

From (3), (4), using Bernoulli inequality, we obtain

$$\frac{|R'''(n,l,m)|}{|R''''(n,l,m)|} = \frac{\left(2^{n+1} \cdot \left(2^{n+1}-1\right) \times \ldots \times \left(2^{n+1}-m+1\right)\right)^l}{\left(2^{n+1}\right)^{lm}} =$$

$$= \prod_{i=0}^{m-1} \left(1 - \frac{i}{2^{n+1}}\right) \geq \prod_{i=0}^{m-1} \left(1 - \frac{m-1}{2^{n+1}}\right) \geq 1 - \frac{m(m-1)}{2^{n+1}} \to 1,$$

as $m^2 = o(2^n)$ $(n \to \infty)$. Thus, if $m^2 = o\ (2^n)$, then any assertion concerning the almost all systems of set $R''''(n,l,m)$ is true for almost all systems of $R'''(n,l,m)$.

We consider $R''''(n,l,m)$ as a space of events, where every event $S \in R''''(n,l,m)$ takes place with the probability $1/R''''(n,l,m) = 2^{-(n+1)ml}$. Consider the random value $\xi_S(\tilde{\alpha})$, which is connected with $S \in R''''(n,l,m)$ as follows: $\xi_S(\tilde{\alpha}) = 1$, if $\tilde{\alpha}$ is the solution of the system S, otherwise $\xi_S(\tilde{\alpha}) = 0$.

From the definition and properties of a linear function it follows that the number of equations of the form $L_i^1 \& L_i^2 \& ... \& L_i^m = 0$, for which $\tilde{\alpha}$ is a solution, is equal to $\left(2^{(n+1)m} - 2^{nm}\right)^l = 2^{nml}\left(2^m - 1\right)^l$.

From this and (4) it follows that $P(\xi_S(\tilde{\alpha}) = 1) = (1 - 1/2^m)^l$.

Consider another random value $\nu = \sum_{\tilde{\alpha} \in B^n} \xi_S(\tilde{\alpha})$. Random value ν has a binomial distribution, because

$$p(\nu = j) = C_{2^n}^j \left(1 - \frac{1}{2^m}\right)^{lj} \left(1 - \left(1 - \frac{1}{2^m}\right)^l\right)^{2^n - j} .$$

Hence, $M\nu = 2^n(1 - 1/2^m)^l$ and $D\nu = 2^n(1 - 1/2^m)^l \cdot (1 - (1 - 1/2^m))^l$, where $M\nu$ and $D\nu$ are the random value ν mathematical expectation and dispersion respectively.

Let $n - \ell(m - \log(2^m - 1)) \underset{n\to\infty}{\longrightarrow} \infty$. It means that $M\nu = 2^n(1 - 1/2^m)^l \underset{n\to\infty}{\longrightarrow} \infty$. Using the Chebishev's second inequation when $t = M\nu/g(n)$, where $g(n)$ is an arbitrary function, satisfying the conditions $g(n) \underset{n\to\infty}{\longrightarrow} \infty$, $g^2(n) = o\ (M\nu)$, we obtain $P\left(|\nu - M\nu| > \dfrac{M\nu}{g(n)}\right) < \dfrac{g^2(n)}{M\nu} \to 0$ as $n \to \infty$. Hence and from the definition of random value ν it follows that almost all systems of the set $R''''(n,l,m)$ have a number of solutions, which asymptotically equals $M\nu$. Therefore, under the conditions of the theorem, almost all systems of $R(n,l,m)$ have also number of solutions asymptotically equal $M\nu = 2^n(1 - 1/2^m)^l$.

The first statement of the Theorem is proved.

Let $n-\ell(m-\log(2^m-1))\underset{n\to\infty}{\longrightarrow}-\infty$. Then $Mv=2^n\left(1-1/2^m\right)^l\underset{n\to\infty}{\longrightarrow}0$. Using Chebishev's first inequation when $t=l$, we obtain $P\left(v\geq 1\right)\underset{n\to\infty}{\longrightarrow}0$ and, therefore, $P\left(v=0\right)\underset{n\to\infty}{\longrightarrow}1$. Hence, it follows that almost all systems S of the set $R''''(n,l,m)$ have no solution. Therefore, when $m^2=o\ (2^n)$, $l^2=o\ (c\,(n,m))$ the second statement of the Theorem is proved.

It is easy to see that it is enough to require $m^2=o\ (2^n)$, because for the greater values of the parameter l the statement of the Theorem also is true (the number of solutions of the system does not increase together with the number of equations).

Now let $n-\ell(m-\log(2^m-1))$ is bounded, as $n\to\infty$. Then $Mv=2^n\left(1-1/2^m\right)^l$ is also bounded as $n\to\infty$. Using the inequations 6 and 5, we obtain

$$P\left(v>r\right)=\sum_{i=0}^{2^n-r}c\left(n,r+i\right)\left(1-\frac{1}{2^m}\right)^{l(r+i)}\left(1-\left(1-\frac{1}{2^m}\right)^l\right)^{2^n-r-i}<$$

$$<c\left(n,r\right)\left(1-\frac{1}{2^m}\right)^{lr}\left(1-\left(1-\frac{1}{2^m}\right)^l\right)^{2^n-r}\left(r+1\right)\times$$

$$\times\left(1-\left(1-\frac{1}{2^m}\right)^l\right)\bigg/\left(r+1-\left(2^n+1\right)\left(1-\frac{1}{2^m}\right)^l\right)<\left(\frac{eMv}{r}\right)\to 0,$$

as $r\to\infty$, because Mv is bounded. Therefore, for almost all systems of the set $R''''(n,l,m)$ and, so, for almost all systems of the set $R(n,l,m)$ the third statement of the Theorem holds. □

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