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Mathematics

DYNAMICAL SAMPLING WITH MOVING DEVICES

A. L. PETROSYAN *

Department of Mathematics, Vanderbilt University, Nashville, TN, USA
Institute of Mathematics of NAS of the Republic of Armenia

The dynamical sampling problem is a new problem in sampling theory dealing with reconstruction of a function from its spatio-temporal samples. The question of reconstructing the signal, when the positions of measuring devices or sensors are not changing over time has been studied earlier. The focus of this paper is on the case when devices are allowed to move. This, for example, may happen when devices are placed on moving cars and measure the air pollution as they travel within a polluted area.

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Introduction. The classical problem in sampling theory is to reconstruct a function from its values (samples) on some subset of its domain. Sometimes the sampling set is small (when measuring devices are expensive or only a few are available) and the samples taken at one time are not enough to do full reconstruction of the function. But if the function, as the initial state of some system, is evolving over time, we can take more samples at different time instances, and try to combine those spatio-temporal samples to recover the function.

Let $f \in V,\ f : X \to \mathbb{C}$ be an unknown function from a class of functions V defined on the set X. Suppose f is evolving over time according to the rule

$$f_i = A^i f, \tag{1}$$

where A is a known operator on V. At time i $(i = 0, \dots, L-1)$ sensors are taking samples at (each time possibly different) locations $\Omega_i \subseteq X$:

$$y_0 = f\Big|_{\Omega_0},\ y_1 = (Af)\Big|_{\Omega_1},\ \dots\ y_{L-1} = (A^{L-1}f)\Big|_{\Omega_{L-1}}. \tag{2}$$

* E-mail: arm.petros@gmail.com

The main problem in dynamical sampling is to uniquely reconstruct the function $f \in V$ from the samples (2).

When $X = \mathbb{Z}_d$, $\Omega_0 = \cdots = \Omega_{L-1} = m\mathbb{Z}_d$, $V = l^2(\mathbb{Z}_d)$ and the operator A is given as a convolution with some kernel, has been addressed in [1]. For the case of shift invariant spaces with $X = \mathbb{R}$, $\Omega_0 = \cdots = \Omega_{L-1} = m\mathbb{Z}$, $V = V(\phi)$, $V(\phi)$ being the space generated by the integer shifts of a function $\phi(x)$ and the operator A is given as a convolution with some function, see [2–4]. In [5] the case when $\Omega_0 \supseteq \Omega_1 \supseteq \cdots \supseteq \Omega_{L-1}$ is treated.

In [6] authors allow also the operator A to be unknown. A generalization of Prony's method to reconstruct the spectrum of A and eventually the function itself is used.

We consider the case when the set X is finite and $V = l^2(X)$. Without loss of generality X can be assumed to be the set $\mathbb{Z}_d = \{1, \ldots, d\}$, V be the $l^2(\mathbb{Z}_d) \cong \mathbb{C}^d$ and the operator A be given as a $d \times d$ matrix.

Later we will also assume that there are fixed M number of measuring devices, i.e., $|\Omega_i| = M$ for every i, and we are allowed to move them anywhere in X after each measurement. For the classical sampling problem with moving devices in the class of bandilimited functions see [7].

Definition. We say that a system of subsets of $X(\Omega_0, \ldots, \Omega_{L-1})$ allows recovery, if any function f can be uniquely reconstructed from samples (2).

When $|\Omega_i| = 1$ for every i, we call the system a path.

Lemma. $(\Omega_0, \ldots, \Omega_{L-1})$ allows recovery, if and only if the system of vectors

$$\{(A^*)^i e_j\}_{j \in \Omega_i,\, i=0,\ldots,L-1} \tag{3}$$

is complete in $\mathbb{C}^d$, where A^* is the adjoint of A and $\{e_j\}_{j=1,\ldots,d}$ is the canonical basis of $\mathbb{C}^d$.

Proof. Note that

$$A^i f(j) = (A^i f, e_j) = (f, (A^*)^i e_j)$$

and these inner products uniquely determine the f, if and only if the system (3) is complete in $\mathbb{C}^d$. □

To find f from the samples (2), the transposes of vectors in (3) (as rows) can be arranged into a big matrix $\mathcal{A} = \mathcal{A}(\Omega_0, \ldots, \Omega_{L-1})$ of size $(|\Omega_0| + \cdots + |\Omega_{L-1}|) \times d$. Then from

$$\mathcal{A} f = y, \tag{4}$$

where y is the vector of sampled values, f can be found by taking the left inverse of $\mathcal{A}$. The left inverse exists, if and only if $\mathcal{A}$ has full row rank or, equivalently, the system (3) is complete in $\mathbb{R}^d$. Note that, for the reconstruction to be stable, the system (3) needs to form a frame for $\mathbb{C}^d$. In finite dimensional space being a frame is equivalent to being complete, but good frame bounds will provide a good condition number for the matrix A.

If (3) is not complete, then we can take the Moore-Penrose pseudoinverse of $\mathcal{A}$ instead. Let $\mathcal{V}$ be the set of all f such that

$$f|_{\Omega_0} = y_0,\ (Af)|_{\Omega_1} = y_1,\ \dots\ (A^{L-1}f)|_{\Omega_{L-1}} = y_{L-1},$$

where y_i is the vector of sampled values at time i at locations Ω_i. This is an affine subspace of $\mathbb{R}^n$. Then the solution to (4), found by taking the Moore–Penrose pseudoinverse of $\mathcal{A}$, is the vector in $\mathcal{V}$ closest to the origin.

Main Theorems. Because we will be working with the A^* mainly, let's drop the $*$. It is the same as assuming the evolution in (1) is given by A^* instead of A.

Theorem 1. For given M there is a system $(\Omega_0,\dots,\Omega_{L-1})$, with $|\Omega_i| = M$ $i = 0,\dots,L-1$, that allows recovery, if and only if

$$LM \geq d \ \text{ and } \ \dim(\ker(A)) \leq M.$$

Proof. The completeness of the system (3) implies that the number of vectors in it should be no less than the dimension of the space $\mathbb{R}^d$, thus $LM \geq d$.

If $\dim(\ker(A)) \leq M$, then in the chain

$$\mathbb{R}^d = Im(A^0) \supseteq Im(A) \cdots \supseteq Im(A^{L-1}) \tag{5}$$

the codimention of each $Im(A^{i+1})$ in $Im(A^i)$ is at most M as

$$\{x \in Im(A^i) | A^{i+1}(x) = 0\} \subseteq \ker(A).$$

Hence, we can select M vectors among $e_1,\dots e_d$ that, if added to $Im(A)$, span the whole $\mathbb{R}^d$. Pick the set of indexes of these vectors to be the Ω_0. If $d \leq 2M$, then there are M vectors among $Ae_1,\dots Ae_d$ that, if added to $\{e_i\}_{i\in\Omega_0}$, span the $\mathbb{R}^d$. Denote the set of indexes of these vectors by Ω_1. Then, for arbitrary choice of the rest of the sets Ω_i, $i \geq 2$, any function can be uniquely recovered by its samples on Ω_i, $i = 0,\dots\Omega_{L-1}$. If $d > 2M$, pick M vectors $\{Ae_i\}_{i\in\Omega_1}$ such that $\{e_i\}_{i\in\Omega_0} \cup \{Ae_i\}_{i\in\Omega_1}$ is a linearly independent system and, if $\{e_i\}_{i\in\Omega_0} \cup \{Ae_i\}_{i\in\Omega_1}$ is added to $Im(A^2)$, it spans the $\mathbb{R}^d$. Because $LM \geq d$, if we continue choosing the sets Ω_i as above, at some step the vectors found this way will span the $\mathbb{R}^d$, and so by Lemma 1, the samples taken at corresponding locations Ω_i will allow recovery.

If $\dim(\ker(A)) > M$, then the system (3) cannot be complete in $\mathbb{R}^d$ as $Im(A)$ will have dimension less than $d - M$. $\square$

When there is only one device taking samples along paths of length $L = d$, is of particular interest. In this case, by Lemma , a path $(j_0,\dots,j_{d-1})$ allows recovery, if and only if the system $\{A^i e_{j_i}\}_{i=0}^{d-1}$ is a basis in $\mathbb{R}^d$ or, equivalently, the matrix $\mathcal{A} = \mathcal{A}(j_0,\dots,j_{d-1})$ in (4) is non-singular.

Theorem 2. For a.e. matrix $A \in M_d(\mathbb{C})$ every path of length d allows recovery.

Proof. From (4), the path $(j_0,\dots,j_{d-1})$ does not allow recovery, if and only if for $\mathcal{A} = \mathcal{A}(j_0,\dots,j_{d-1})$,

$$\det(\mathcal{A}) = 0.$$

The set of matrices, for which this holds, is a lower dimensional algebraic variety in the space of all $d \times d$ matrices. Any lower dimensional algebraic variety has measure zero, and there are only finite number of paths of length d. $\square$

Theorem 3. For a given matrix A, there is a path that allows recovery, if and only if in the Jordan form of A there is at most one Jordan block corresponding to 0-eigenvalue. If the size of that Jordan block is κ, then there are at least $(d-\kappa)!$ paths of length d that allow recovery.

Proof. For any square matrix, number of its Jordan blocks corresponding to 0-eigenvalue is equal to the dimension of its kernel. So, if for matrix A there is a path that allows recovery, then from the previous theorem A can have at most one such Jordan block. Hence in (5) $\dim(Im(A^{i-1})) - \dim(Im(A^i))$ is 1 for $i = 1, \ldots, \kappa$, and is 0 for $i = \kappa+1, \ldots, d-1$.

Following the proof of Theorem 1, there is at least one way to select the vector e_{j_0} such that it is not in $Im(A)$. After e_{j_0} has been selected, there is at least one way to choose the vector Ae_{j_1} that is not in $Im(A^2)$. The same way there is at least one way to choose the vectors in

$$e_{j_0}, \ldots, A^{\kappa-1}e_{j_{\kappa-1}}$$

such that each $A^{i-1}e_{j_{i-1}}$ is not in $Im(A^i)$ $i = 1, \ldots, \kappa$.

Because the codimension of $Im(A^{\kappa+1})$ in $Im(A^\kappa)$ is 0, there are at least $d-\kappa$ ways to select the vector $A^\kappa e_{j_\kappa}$. After we choose it, there are at least $d-\kappa-1$ ways we can choose the $A^{\kappa+1}e_{j_{\kappa+1}}$ so that the system $e_{j_0}, \ldots, A^{\kappa-1}e_{j_{\kappa+1}}$ stays independent. Then, for the $A^{\kappa+2}e_{j_{\kappa+2}}$, there will be at least $d-\kappa-2$ ways to choose it such the system stays independent and so on. This gives a lower bound on the number of paths that allow recovery as $(d-\kappa)!$. $\square$

Note that, for matrices of the form

$$A = \left(\begin{array}{cccccccc}
0 & a_{12} & a_{12} & \ldots & a_{1\kappa} & & & \\
 & 0 & a_{23} & \ldots & a_{2\kappa} & & & \\
 & & \ddots & & \vdots & \text{\Large 0} & & \\
 & & & 0 & a_{\kappa-1,\kappa} & & & \\
 & & & & 0 & & & \\
 & & & & & a_{\kappa+11,\kappa+1} & & \\
 & \text{\Large 0} & & & & \ddots & & \\
 & & & & & & & a_{d,d}
\end{array}\right),$$

where $a'_{i,i+1} \neq 0$ $i = 1, \ldots, \kappa-1$ and $a'_{i,i} \neq 0$ $i = \kappa+1, \ldots, d$, there are exactly $(d-\kappa)!$ paths that allow recovery.

Theorem 4. For any non-singular matrix A there are at least $d!$ paths that allow recovery. Moreover, there are exactly $d!$ such paths, if and only if A is a diagonal matrix up to permutation of rows.

Proof. Suppose the number of paths allowing recovery is $d!$. Then, from the proof of the previous theorem, for every e_i there is an e_j such that e_i and Ae_j are linearly dependent:

$$Ae_j = ce_i$$

for some $c \in \mathbb{C}$.

Let $e_{i_1}, \dots e_{i_m}$ be a chain of vectors such that

$$Ae_{i_{s+1}} = c_s e_{i_s},\ s = 1, \dots m-1 \text{ and } Ae_{i_1} = c_m e_{i_m}.$$

Consider the matrix $E_{i_1 i_m}A$, where $E_{i_1 i_m}$ permutes the i_1-th and i_m-th rows of a matrix when multiplied from the left. Then

$$E_{i_1 i_m} A e_{i_1} = c_m E_{i_1 i_m} e_{i_m} = c_m e_{i_1},$$

$$E_{i_1 i_m} A e_{i_{s+1}} = c_s E_{i_1 i_m} e_{i_s} = c_s e_{i_s},\ s = 2, \dots m-1,$$

and

$$E_{i_1 i_m} A e_{i_2} = c_1 E_{i_1 i_m} e_{i_1} = c_1 e_m.$$

Also for every e_i, which is not in the chain,

$$E_{i_1 i_m} A e_i = c E_{i_1 i_m} e_j = c e_j = A e_i.$$

Thus, we were able to reduce the number of elements in the chain by one. If the process of permuting the columns of A is done for every such chain in the same way as above, we will eventually end up with a permuted version $\bar{A}$ of A such that for every i

$$\bar{A} e_i = \lambda_i e_i,$$

which implies that $\bar{A}$ is diagonal. □

C o r o l l a r y . A non-singular self adjoint matrix has $d!$ paths of length d that allow recovery, if and only if it is diagonal.

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