PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2014, № 3, p. 13–15

M a t h e m a t i c s

ON SOLVABILITY OF PSEUDODIFFERENTIAL EQUATIONS IN SPACES WITH QUASIHOMOGENEOUS NORM

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In the article solvability questions for a class of pseudodifferential operators with quasihomogeneous nongenerate on the unit sphere symbol in spaces of anisotropic potentials or, in other words, spaces with quasihomogeneous norm are studied.

MSC2010: 26A33; 35S10.

***Keywords*:** pseudodifferential operators, anisotropic potentials, spaces with quasihomogeneous norm.

Let R_n be the Euclidean space with points $x=(x_1,x_2,...,x_n), r=(r_1,r_2,...,r_n)$ a vector with positive components, $\dfrac{1}{r^*}=\dfrac{1}{n}\sum_1^n \dfrac{1}{r_j}$, and $\lambda=(\lambda_1,\lambda_2,...,\lambda_n)$, where $\lambda_j=\dfrac{r*}{r_j}, j=1,2,...,n$. By $\rho(x)$ we denote the function, positive for $x\neq 0$, defined implicitly by the equality

$$\sum_{i=1}^n x_i^2\rho^{-2\lambda_i}=1.$$

Let I_r be a pseudodifferential operator (ΨDO) with the symbol $\rho^{r^*}(\xi)$, $I^r\varphi=F^{-1}(\rho^{r^*}(\xi)F\varphi(\xi))$, where F and F^{-1} are direct and inverse Fourier transforms. If $r_1=r_2=...=r_n=0$, then $I_r=I$ is the identity operator. When $0<-r^*<n$, the operator I_r is called anisotropic (Riesz) potential.

It is natural (see [1]) to denote the completion of $C_0^\infty(R^n)$ in the norm

$$\|f\|=\|F^{-1}(\rho^{r^*}(\xi)F\varphi(\xi))\|_p,\quad 1<p<\infty,$$

by the symbol $\dot{\mathrm{w}}_\mathrm{p}^\mathrm{r}$ (see also $[2,3]$) and call it a space with quasihomogeneous norm or space of anisotropic potentials. If $r^*<n/p$, then $\dot{\mathrm{w}}_\mathrm{p}^\mathrm{r}$ is the space of functions representable by anisotropic potentials. When $r^*\geq n/p$ the space $\dot{\mathrm{w}}_\mathrm{p}^\mathrm{r}$ is no longer a function space; its elements are classes, in which the functions that differ by corresponding polynomials are identified $[1,4]$.

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When $r_1 = r_2 = ... = r_n = 0$, we set $\dot{w}_p^r = L_p(R_n)$. The space $\dot{w}_p^{-r}$ is defined as the dual of $\dot{w}_p^r$.

Suppose that the function $K(\xi)$ is λ-homogeneous of degree s, $-\infty < s < \infty$, i.e. for every $t > 0$ and arbitrary $\xi \neq 0$, $K(t^\lambda \xi) \equiv K(t^{\lambda_1}\xi_1, ..., t^{\lambda_n}\xi_n) = t^s K(\xi)$. As is shown in [1], continuous for $\xi \neq 0$, λ-homogeneous $K(\xi)$ function is the symbol of a bounded operator

$$K : \dot{w}_2^r \quad \to \quad \dot{w}_2^{\kappa r},$$

where $\kappa = 1 - s/r^*$ and $\kappa r = (\kappa r_1, ..., \kappa r_n)$.

Let $\Psi = (\Psi_{jk})$, $j = 0,1,...,M-1$, $k = 0,1,...,N-1$, be the matrix of ΨDO Ψ_{jk} with the symbol $\Psi_{jk}(\xi)$ continuous for $\xi \neq 0$ and λ-homogeneous of degree $\alpha_j - \beta_k$:

$$\Psi_{jk}(t^{\lambda_1}\xi_1, ..., t^{\lambda_n}\xi_n) = t^{\alpha_j - \beta_k}\Psi_{jk}(\xi), \quad \text{for } t > 0, \ -\infty < \alpha_j, \beta_k < \infty,$$

$$u = (u_0, u_1, ..., u_{N-1}), \ \ f = (f_0, f_1, ..., f_{M-1}).$$

Now we consider the solvability of the system of equations

$$\Psi u = f \quad \text{in} \quad \dot{w}_2^r.$$

Theorem. The ΨDO

$$\Psi : \prod_{k=0}^{N-1} \dot{w}_2^{(1-\beta_k/r^*)r} \quad \to \prod_{j=0}^{M-1} \dot{w}_2^{(1-\alpha_j/r^*)r}$$

is bounded and left-invertible (right-invertible) if and only if rank$(\Psi_{jk}(\xi)) = N$ (respectively M) for all $\xi \neq 0$.

Proof. The case $M = N = 1$ of the Theorem was proved in [5].

Denot by $Q(\xi)$ the matrix $(Q_{jk}(\xi))$ with elements

$$Q_{jk}(\xi) = \rho^{\beta_k - \alpha_j}\Psi_{jk}(\xi), \ j = 0,1,...,M-1; k = 0,1,...,N-1,$$

and by $I_1(\xi)\,\big(I_2(\xi)\big)$ the diagonal $M \times M$ $(N \times N)$ matrix with $\rho^{r*-\alpha_j}(\xi)$ $\big($respectively $\rho^{\beta_k - r^*}(\xi)\big)$ elements on the main diagonal. The matrices $Q(\xi), I_1(\xi), I_2(\xi)$ are the symbols of bounded operators Q, I_1, I_2 satisfaying

$$Q = I_1 \Psi I_2 : \prod_{k=0}^{N-1} L_2(R_n) \quad \to \prod_{j=0}^{M-1} L_2(R_n).$$

But the operators I_1 and I_2 are isomorphisms between the appropriate spaces with quasihomogeneous norms, so, Ψ is left invertible (right invertible) if and only if Q has left (right) inverse and rank $\Psi(\xi)$= rank $Q(\xi)$ for all $\xi \neq 0$.

Suppose rank $Q(\xi) = N$ for each $\xi \neq 0$ (and so $N = M$).

Set $K(\xi) = (Q^*(\xi)Q(\xi))^{-1}Q^*(\xi)$, where $Q^*(\xi)$ is the conjugate transposition of $Q(\xi)$. Then the elements of $K(\xi)$ are continuous for $\xi \neq 0$, λ-homogeneous of degree 0 and $K(\xi)Q(\xi) = I$. So, $K(\xi)$ is the symbol of a bounded left inverse of Q.

Conversely, if rank$Q(\eta) < N$ for some $\eta \neq 0$, i.e. there exists a unit vector $c = (c_0, ..., c_{N-1})$ such that $Q(\eta)c = 0$. Take $\varphi \in C_0^\infty(R^n)$ with $\varphi(\xi) \geq 0$ for all ξ, $\varphi(\xi) = 0$ for $|\xi| > 1$ and $\|\varphi\|_2$= 1.

For $\varepsilon > 0$, it is set $g_\varepsilon = (g_{\varepsilon 0}, ..., g_{\varepsilon k-1})$, $g_{\varepsilon k}(\xi) = \varepsilon^{\frac{n}{2}} \varphi\Big(\dfrac{\xi - \eta}{\varepsilon}\Big) c_k$, where $k = 0, ..., N-1$. Then

$$\left\| g_\varepsilon, \prod_{k=0}^{N-1} L_2 \right\|^2 = \sum_{k=0}^{N-1} \|g_{\varepsilon k}\|_2^2 = \sum_{k=0}^{N-1} c_k^2 \varepsilon^{-n} \int_{R^n} \left| \varphi\Big(\frac{\xi - \eta}{\varepsilon}\Big) \right|^2 d\xi = 1.$$

Let f_ε be the inverse Fourier transform of g_ε. Then, by Placherel's theorem,

$$\left\| f_\varepsilon, \prod_{k=0}^{N-1} L_2 \right\| = 1.$$

But

$$\left\| Qf_\varepsilon, \prod_{j=0}^{M-1} L_2 \right\|^2 = \sum_{j=0}^{M-1} \int_{|\xi - \varepsilon| < \varepsilon} \varepsilon^{-n} \left| \sum_{k=0}^{N-1} Q_{jk}(\xi) \varphi\Big(\frac{\xi - \eta}{\varepsilon}\Big) c_k \right|^2 d\xi \le$$

$$\le \sum_{j=0}^{M-1} \sup_{|\xi - \eta| < \varepsilon} \left| \sum_{k=0}^{N-1} Q_{jk}(\xi) c_k \right|^2 \quad \to \quad 0 \quad \text{as} \quad \varepsilon \to 0,$$

since $Q_{jk}(\xi)$ are continuous in η and $Q(\eta)c = 0$. This disrupts the criterion of the existence of a left inverse operator, so, Q is not left invertible.

Since Q^* the adjoint of Q is a ΨDO with the symbol $Q^*(\xi)$ it has left inverse if and only if rank $Q(\xi) = M$.

To complete the proof of Theorem it remains to use duality. □

Received 16.07.2014

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