PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2018, **52**(2), p. 88–92

Mathematics

ON A QUESTION OF A. SOZUTOV

N. E. MIRZAKHANYAN[1] *, H. V. PILIPOSYAN [2]**

[1] *Chair of Algebra and Geometry YSU, Armenia*
[2] *Chair of Mathematical Cybernetics RAU, Armenia*

In the paper an answer to a problem posed by A.I. Sozutov in the Kourovka Notebook is given. The solution is based on some modification of the method that was proposed for constructing a non-abelian analogue of the additive group of rational numbers, i.e. a group whose center is an infinite cyclic group and any two non-trivial subgroups of which have a non-trivial intersection.

MSC2010: Primary 20F05; Secondary 20E22.

***Keywords*:** central extension, periodic group, normal closure.

Introduction. It is well known that if in an abelian group any two non-trivial subgroups have a non-trivial intersection, then this group is locally cyclic and, therefore, it is isomorphic to a quotient group of the additive group of rational numbers. First examples of *non-abelian* groups, in which any two non-trivial subgroups have a non-trivial intersection, were constructed in [1] (the solution of the problem of P.G. Kontorovich see [2], *Q*. 1.63). Constructed non-abelian analogs of the group of rational numbers, denoted by $A(m,n)$ are central extensions of the free Burnside group $B(m,n)$ with an infinite center generated by a new generating element d of infinite order. Recall that the free Burnside group $B(m,n)$ of period n and rank m has the following presentation

$$B(m,n) = \langle a_1, a_2, \dots, a_m \mid X^n = 1 \rangle,$$

where X runs through the set of all words in the alphabet $\{a_1^{\pm 1}, a_2^{\pm 1}, \dots, a_m^{\pm 1}\}$. If we add one more generation d to the set of generators $B(m,n)$, which commutes with each generator a_i of $B(m,n)$, $i = 1,2,\dots,m$, and if we replace the relations $\{A^n = 1 | A \in \mathcal{E}\}$ by $\{A^n = d | A \in \mathcal{E}\}$, then we get the group $A(m,n)$.

If to the defining relations of the group $A(m,n)$ add another defining relation $d^k = 1$, then in the obtained group $A'(m,n)$ the center generated by d will have order k. Group $A'(m,n)$ has an interesting property: group $A'(m,n)$ *admits only the discrete topology*. The existence of an untopologizable countable group was

* E-mail: mnarine@ysu.am **E-mail: haykpiliposyan94@gmail.com

posed by A.A. Markov and remained open for several decades. Among the various applications of groups $A(m,n)$ note also the recent work [4], where was used the groups $A(m,n)$ for a description of $\{2,3\}$-group, which act freely on some non-trivial abelian group. We also note the paper [5], where some automorphisms of the groups $A_{\mathcal{D}}(m,n)$ are investigated (the definition of $A_{\mathcal{D}}(m,n)$ see below).

A modification of the definition of the groups $A(m,n)$ allows to construct a group, whose center coincides with a given abelian group $\mathcal{D}$, and the factor group by subgroup $\mathcal{D}$ is isomorphic to the free Burnside group $B(m,n)$ of an arbitrary fixed rank $m > 1$.

We proceed to precise definitions. Let the integer $m > 1$ and odd $n \geq 665$ be fixed. Consider the set of elementary words

$$\mathcal{E} = \bigcup_{\alpha=1}^{\infty} \mathcal{E}_{\alpha}. \tag{1}$$

which is defined in [3, VI.2.1]. The set $\mathcal{E}$ is countable (Theorem 2.13 of Chap. VI [3]), that is, its element can be numbered by natural numbers. We fix some numbering and let $\mathcal{E} = \{A_j | j \in \mathbb{N}\}$ ($\mathbb{N}$ is the set of all natural numbers).

We also fix an arbitrary at most countable abelian group $\mathcal{D}$, given by the generators and defining relations:

$$\mathcal{D} = \langle d_1, d_2, \ldots, d_i, \ldots \mid r = 1, \;\; r \in \mathcal{R} \rangle, \tag{2}$$

where $\mathcal{R}$ is some set of words in the group alphabet $d_1, d_2, \ldots, d_i, \ldots$

Denote by $A_{\mathcal{D}}(m,n)$ the group given by the system of generators two kinds

$$a_1, a_2, \ldots, a_m \tag{3}$$

and

$$d_1, d_2, \ldots, d_i, \ldots, \tag{4}$$

and the system of defining relations:

$$r = 1 \;\text{ for all } r \in \mathcal{R}, \;\; a_i d_j = d_j a_i, \tag{5}$$

$$A_j^n = d_j \;\text{ for all }\; A_j \in \mathcal{E}, \;\; i = 1, 2, \ldots, m \;\text{ and }\; j, k \in \mathbb{N}. \tag{6}$$

We note, that if as the group $\mathcal{D}$ we take the infinite cyclic group

$$\mathcal{D}_0 = \langle d_1, d_2, \ldots, d_i, \ldots \mid d_j d_k^{-1} = 1, \;\; j, k \in \mathbb{N} \rangle, \tag{7}$$

then the obtained group $A_{\mathcal{D}_0}(m,n)$ will exactly coincide with the group $A(m,n)$.

From the relations (6) follows that the groups $A_{\mathcal{D}}(m,n)$ are m-generated groups with generators (3). For groups $A_{\mathcal{D}}(m,n)$ for any $m > 1$ and odd $n \geq 665$ and for any abelian group $\mathcal{D}$ (2) the following assertions hold.

Proposition A. (see [6]) .

1. In the group $A_{\mathcal{D}}(m,n)$ the identity holds $[x^n, y] = 1$.

2. The verbal subgroup of $A_{\mathcal{D}}(m,n)$ corresponding to the word x^n coincides with abelian group $\mathcal{D}$.

3. The center of $A_{\mathcal{D}}(m,n)$ coincides with $\mathcal{D}$.

4. The factor group of the group $A_{\mathcal{D}}(m,n)$ with respect to the subgroup $\mathcal{D}$ is the free Burnside group $B(m,n)$.

We draw the attention of the reader to a certain freedom in the construction of groups $A_{\mathcal{D}}(m,n)$. First, there is a certain arbitrariness in the order of the numbering of elementary periods $A_j \in \mathcal{E},\ j \in \mathbb{N}$. Second, we have a large degree of freedom when choosing the presentation (2) of the abelian group $\mathcal{D}$. Thus, by the defining relations of the form (6), for a fix group $\mathcal{D}$ we will get different groups $A_{\mathcal{D}}(m,n)$. In this case, assertions 1–4 hold for each of them.

Sozutov in the Kourovka Notebook posed the question (*Q*. 18.94 [2]): *Let G be a group without involutions, a be an element of it that is not a square of any element of G and k be an odd positive integer. Is it true that the quotient* $G/\langle (a^k)^G \rangle$ *does not contain involutions*.

Using Proposition A, we construct a counter-example to the indicated problem of A.L. Sozutov. Consider the direct product of an infinite cyclic group with an additive group of rational numbers:

$$\mathbb{Z} \times \mathbb{Q} = \langle d_1, d_2 \ldots, d_i, \ldots \mid d_i^{i-1} = d_{i-1},\ i \geq 3 \rangle. \tag{8}$$

Theorem. There exists a central extension G of direct product $\mathbb{Z} \times \mathbb{Q}$ by a free Burnside group such that G does not contain involutions, an element $a_1 \in G$ is not square of any element of G and any equation of the form $x^k = 1$ $(k > 1)$ in the factor group $G/\langle (a_1^3)^G \rangle$ has a non trivial solution.

Proof of the Main Result. To simplify the proof of the Theorem suppose that $n \geq 665$ is an odd number that is divisible by 3.

The following lemma follows directly from the definition of the notion of elementary period.

Lemma 1. A word in the group alphabet $a_2, \ldots, a_m$ is an elementary period of some rank α if and only if it is an elementary period of rank α among the words in the group alphabet $a_1, a_2, \ldots, a_m$.

Let $m \geq 3$. We denote by $\mathcal{E}(a_1, \ldots, a_m)$ and $\mathcal{E}(a_2, \ldots, a_m)$ the set of elementary words (1) in the alphabets $a_1, \ldots, a_m$ and $a_2, \ldots, a_m$ respectively. From Lemma 1 and from the definition (1) it follows that

Lemma 2. The sets $\mathcal{E}(a_1, \ldots, a_m)$ and $\mathcal{E}(a_2, \ldots, a_m)$ can be constructed so that $\mathcal{E}(a_1, \ldots, a_m) \supset \mathcal{E}(a_2, \ldots, a_m)$.

By the Lemma 2 the set $\mathcal{E}(a_1, \ldots, a_m)$ can be represented as a disjoint union $\mathcal{E}(a_1, \ldots, a_m) = \mathcal{E}' \cup \mathcal{E}(a_2, \ldots, a_m)$, where $\mathcal{E}'$ contains those elementary periods, in which the letter a_1 participates. Further suppose that, the elements of $\mathcal{E}'$ are numbered by odd natural numbers, and the elements of $\mathcal{E}(a_2, \ldots, a_m)$ are numbered by even numbers. The words $a_1^{\pm 1}$ and $[a_1^3, a_2] = a_1^3 a_2 a_1^{-3} a_2^{-1}$ are obviously elementary periods of rank 1 and are not conjugate in rank 0, therefore they can be included in $\mathcal{E}'$. For definiteness, we denote $A_1 = a_1$ and $A_3 = [a_1^3, a_2]$.

Now adding the relation $d_2 = 1$ to the defining relations of the group $\mathbb{Q}$, we construct its factor group $\mathbb{Q}_1$:

$$\mathbb{Q}_1 = \langle d_2, \ldots, d_i, \ldots \mid d_2 = 1, d_i^i = d_{i-1},\ \ i \geq 3 \rangle. \tag{9}$$

We construct two groups $A_D(m,n)$ for the groups $D = \mathbb{Q}_1$ and the direct product $D = \mathbb{Z} \times \mathbb{Q}$ (8) by the scheme (3)–(6).

We define the first group in the following way:

$$A_{\mathbb{Q}_1}(m-1,n) = \langle a_2,\dots,a_m \mid A_{2j}^n = d_{j+1},\ \ j \geq 1\rangle. \tag{10}$$

From Proposition A and the definition of the group $\mathbb{Q}_1$ it follows that the center of the group $A_{\mathbb{Q}_1}(m-1,n)$ is the group $\mathbb{Q}_1$. If in every word A_{2k-1}^n, where $A_{2k-1} \in \mathcal{E}'$ and $k \geq 3$, remove all occurrences of a_1, then we obtain a certain word in the alphabet $a_2,\dots,a_m$, which (by Proposition A) in group $A_{\mathbb{Q}_1}(m-1,n)$ is equal to some element z_k from the center $\mathbb{Q}_1$. We fix this element z_k and construct second group $A_{\mathbb{Z}\times\mathbb{Q}}(m,n)$:

$$\begin{aligned} &A_{\mathbb{Z}\times\mathbb{Q}}(m,n) = \\ &= \langle a_1,a_2,...,a_m \mid A_1^n = d_1,\ A_3^n = d_2,\ A_{2k-1}^n = z_k, k \geq 3, A_{2j}^n = d_{j+1},\ j \geq 1\rangle. \end{aligned} \tag{11}$$

L e m m a 3. Every finite subgroup of $A_{\mathbb{Z}\times\mathbb{Q}}(m,n)$ is contained in some cyclic subgroup of order n, and the order of any element is either infinite or divides n.

P r o o f . Replacing the group $A(m,n)$ by $A_{\mathbb{Z}\times\mathbb{Q}}(m,n)$, repeating the arguments of [5], Chapter VII, we verify that every finite subgroup of the group $A_{\mathbb{Z}\times\mathbb{Q}}(m,n)$ is contained in some cyclic subgroup of order n, and, hence, the order of any element is either infinite or divides n. □

Thus, by Lemma 3 the group $A_{\mathbb{Z}\times\mathbb{Q}}(m,n)$ does not contain involutions.

Assume that the square of some element Xd is equal to a_1 in $A_{\mathbb{Z}\times\mathbb{Q}}(m,n)$, where X is a word in the alphabet (3) and $d \in \mathbb{Z}\times\mathbb{Q}$. Then $X^2 = a_1$ in $B(m,n)$ and, by virtue of Theorem 3.3 of chapter VI [3], $X = a_1^k$ for some integer k.

From the Proposition A it follows that for any words X and Y in alphabet (3) the equality $X = Y$ is satisfied in $B(m,n)$ if and only if there exists an element $d \in \mathbb{Z}\times\mathbb{Q}$ such that $Xd = Y$ in the group $A_{\mathbb{Z}\times\mathbb{Q}}(m,n)$. Therefore, $a_1^{2k}z^2 = a_1$ in $A_{\mathbb{Z}\times\mathbb{Q}}(m,n)$ for some $z \in \mathbb{Z}\times\mathbb{Q}$. By item 4 of Proposition A, we have $2k-1 = nt$ for some integer t, and by virtue of relations (11) we have $d_1^t = z^{-2}$. Since t is an odd number and $\mathbb{Z}\times\mathbb{Q}$ is the group without torsion, the equality $d_1^t = z^{-2}$ in $\mathbb{Z}\times\mathbb{Q}$ is possible only for $z = 1$ and $t = 0$, which leads to the incorrect equality $2k-1 = 0$. Thus, a_1 is not the square of any element.

To complete the proof of the Theorem, it remains to show that in the quotient group $G/\langle\langle a^3\rangle\rangle$, the equation $x^k = 1$ has a nontrivial solution for any natural number $k > 1$ where $G = A_{\mathbb{Z}\times\mathbb{Q}}(m,n)$ and $\langle\langle a^3\rangle\rangle$ is the normal closure of the element a_1^3 in G.

A direct calculation shows that the groups $G/\langle\langle a^3\rangle\rangle$ and $A_{\mathbb{Q}_1}(m-1,n)$ have the same sets of generators and defining relations, i.e. $G/\langle\langle a^3\rangle\rangle = A_{\mathbb{Q}_1}(m-1,n)$. Subgroup $\mathbb{Q}_1$ of the group $A_{\mathbb{Q}_1}(m-1,n)$ is obtained from the full group of rational numbers by adding defining relation $d_2 = 1$. It is easy to understand that in every quotient group of the additive group of rational numbers obtained by adding one relation, each equation $x^k = 1$ $(k > 1)$ has a nontrivial solution.

Received 04.05.2018

REFERENCES

1. **Adian S.I.** On Some Torsion-Free Groups. // Izv. Akad. Nauk SSSR. Ser. Mat., 1971, v. 35, № 3, p. 459–468.
2. **Mazurov V.D., Merzlyakov Yu.I., Churkin V.A.** The Kourovka Notebook. Unsolved Problems in Group Theory (ed. 8). Novosibirsk: Institute of Math., 2014.
3. **Adian S.I.** The Burnside Problem and Identities in Groups. Berlin-New York: Springer-Verlag, Results in Mathematics and Related Areas, 1979, 95 p.
4. **Zhurtov A.Kh., Lytkina D.V., Mazurov V.D., Sozutov A.I.** On Periodic Groups Acting Freely on Abelian Groups. // Proc. Steklov Inst. Math. (Suppl. 1), 2014, v. 285, p. 209–215.
5. **Stepanyan Sh.A.** On Automorphisms of the Relatively Free Groups Satisfying the Identity $[x^n, y] = 1$. // Proceedings of the YSU. Physical and Mathematical Sciences, 2017, v. 51, № 2, p. 196–198.
6. **Adian S.I., Atabekyan V.S.** Central Extensions of Free Periodic Groups. Emil Artin International Conference Dedicated to the 120th Anniversary of Emil Artin. Yer., 27.05–02.06.2018.