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C O M M U N I C A T I O N S

M a t h e m a t i c s

ON AUTOMORPHISMS AND ENDOMORPHISMS OF CC GROUPS

H. T. ASLANYAN *

Chair of Mathematical Cybernetics RAU, Armenia

We consider the automorphisms description question for the semigroups $\mathrm{End}\, G$ of a group G having only cyclic centralizers (CC) of nontrivial elements. In particular, we prove that each member of the automorphism group $\mathrm{Aut}(G)$ of a group G from this class is uniquely determined by its action on the elements from the subgroup of inner automorphisms $\mathrm{Inn}(G)$. Note that, typical examples of CC groups are absolutely free groups, free periodic groups of large enough odd periods, n-periodic and free products of CC groups.

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Introduction. A group G is said to be a CC group, if the centralizer of each non-trivial element of G is a cyclic group. It is well known that absolutely free groups and free periodic groups of large enough odd periods (see [1]) are CC groups. It is easy to show that the free product of an arbitrary family of CC groups also is a CC group. It follows from Theorem 5 of the paper [2] (see also [3]) that the same is true for n-periodic products of CC groups. Another wide class of CC groups will be considered bellow. In [4] there were constructed first examples of infinite independent systems of group identities to solve the finite basis problem posed by B. Neumann in 1937 well-known in group theory. In the monograph [1] it is proved that for any odd $n \geq 1003$ the following family of two-variable identities

$$\{[x^{pn},\, y^{pn}]^n = 1\}, \tag{1}$$

where the parameter p ranges over all primes, is irreducible, that is none of the identities of this family follows from the others. Therefore, if for a given set of primes $\mathcal{P}$ and for a fixed positive integer $m > 1$ we denote by $\Gamma_m(\mathcal{P})$ a relatively

* E-mail: haikaslanyan@gmail.com

free group of rank m of a variety $\mathbb{A}_{\mathcal{P}}$ defined by all identities of the form (1) for $p \in \mathcal{P}$, then there exist a continuum of varieties and a continuum of non-isomorphic relatively free groups $\Gamma_m(\mathcal{P})$ corresponding to the different sets of primes $\mathcal{P}$. It was proved in [5], that for any rank m and for any set of primes $\mathcal{P}$ the centralizer of any non-identity element of the relatively free group $\Gamma_m(\mathcal{P})$ is a cyclic group, that is each of the groups $\Gamma_m(\mathcal{P})$ is a CC group.

In this paper we consider the question on the description of the automorphisms of $\mathrm{End}(G)$ for a CC group G. The automorphism description question the for $\mathrm{End}(A)$ of a free algebra A in a certain variety was considered by different authors since 2002. The same problem for $\mathrm{End}(F)$, where F is a finitely generated free group, for a free Burnside group of odd period $n \geq 1003$ or a free monoid were solved in [6–8]. A generalization of results from [6, 7] was obtained in [9]. Note that, for instance, finitely generated free periodic groups of period 3 are not CC groups (this case was described in [10].

To formulate the results recall same notations. The group of all inner automorphism of a group G is denoted by $\mathrm{Inn}(G)$. We denote by i_a the inner automorphism of G defined by an element $a \in F$. By definition we have $i_a(x) = axa^{-1}$ for any $x \in G$. Here we investigated a more general situation that was considered in [9]. Our main result is the following theorem.

Theorem. Let Φ be an arbitrary automorphism of the endomorphism semigroup $\mathrm{End}(G)$ of a non-cyclic CC group G. If $\Phi(i_a) = i_a$ for any $i_a \in \mathrm{Inn}(G)$, then $\Phi(\delta) = \delta$ for any endomorphism $\delta \in \mathrm{End}(G)$ whose image $\mathrm{Im}\delta$ is not cyclic.

Corollary. For any $\Phi \in \mathrm{Aut}(\mathrm{Aut}(G))$ of a non-cyclic CC group G such that $\Phi(i_a) = i_a$ for any $i_a \in \mathrm{Inn}(G)$ the equality $\Phi(\delta) = \delta$ holds for all $\delta \in \mathrm{Aut}(G)$.

The Proof of Theorem. We will derive the proof from several lemmas, which will be proved below.

Lemma 1. If G is a CC group, then:

a) any non-trivial element x of G belongs to the unique maximal cyclic subgroup, which is the centralizer of x;

b) if non trivial elements a^m and b^n of G commute, then a and b belong to the same cyclic subgroup.

Proof.

a) Any maximal cyclic subgroup A of G is a subset of the centralizer $C(x)$ of each non-trivial element $x \in A$. Hence $A = C(x)$, because $C(x)$ also is cyclic. Further, any two different maximal cyclic subgroups A and B of G generate a non-cyclic subgroup $gp\{A, B\}$ in the centralizer of each element $y \in A \bigcap B$, so this relation implies the equality $y = 1$, because the centralizer of each non-trivial element is cyclic.

b) Let the cyclic group $gp\{x\}$ be the centralizer of a^m. Then $b^n, a^m \in gp\{x\}$. Since $b^n \in gp\{b\}$, $a^m \in gp\{a\}$ and $gp\{x\}$ is a maximal cyclic subgroup by virtue of a), we get $gp\{b\} \subset gp\{x\}$ and $gp\{a\} \subset gp\{x\}$, since any non-trivial element belongs to the unique maximal cyclic subgroup. In particular, a and b belong to the cyclic group $gp\{x\}$. □

L e m m a 2. For any $a,x \in G$ and $\delta \in \mathrm{End}(G)$ the element $\delta(a)^{-1} \cdot \Phi(\delta)(a)$ belongs to the centralizer of the element $\Phi(\delta)(x)$ and vice versa.

P r o o f . Consider an arbitrary endomorphism $\delta \in \mathrm{End}(G)$, and apply the product $\delta \circ i_a$ of automorphisms to an element $x \in G$. By definition we have

$$(\delta \circ i_a)(x) = \delta(i_a(x)) = \delta(a)\delta(x)\delta(a^{-1}) = (i_{\delta(a)} \circ \delta)(x).$$

Hence, the following equality holds:

$$\delta \circ i_a = i_{\delta(a)} \circ \delta. \tag{2}$$

By condition of Theorem , the restriction of an automorphism Φ from $\mathrm{End}(G)$ to the subgroup $\mathrm{Inn}(G)$ is the identity automorphism, that is,

$$\Phi\big|_{\mathrm{Inn}(G)} = 1_{\mathrm{Inn}(G)}. \tag{3}$$

We will show that equality

$$\delta(a)^{-1} \cdot \Phi(\delta)(a) = 1$$

for any $a \in F$ and $\delta \in \mathrm{End}(G)$, which means that

$$\Phi(\delta) = \delta$$

for any $\delta \in \mathrm{End}(G)$.

Applying the automorphism Φ to both sides of Eq. (2) and taking into account (3), we obtain the equality

$$\Phi(\delta) \circ i_a = i_{\delta(a)} \circ \Phi(\delta). \tag{4}$$

Now applying both sides of Eq. (4) to an arbitrary element $x \in G$, we get

$$\Phi(\delta)(a) \cdot \Phi(\delta)(x) \cdot \Phi(\delta)(a)^{-1} = \delta(a) \cdot \Phi(\delta)(x) \cdot \delta(a)^{-1}. \tag{5}$$

Eq. (5) implies that the element $\delta(a)^{-1} \cdot \Phi(\delta)(a)$ belongs to the centralizer of the element $\Phi(\delta)(x)$ for every $a,x \in G$ and vice versa. $\square$

L e m m a 3. If the image of $\Phi(\delta)$ is not a cyclic group for some endomorphism $\delta \in \mathrm{End}(F)$, then

$$\Phi(\delta) = \delta. \tag{6}$$

P r o o f . Suppose $\Phi(\delta)(x)$ and $\Phi(\delta)(y)$ do not belong to a same cyclic subgroup for some elements $x,y \in G$. Then they belong to different maximal cyclic subgroups, say A and B respectively. By Lemma 2, we have $\delta(a)^{-1} \cdot \Phi(\delta)(a) \in A \bigcap B$ for any element $a \in F$. On the other hand, $A \bigcap B = \{1\}$ by Lemma 1 (because $A \neq B$). Therefore,

$$\delta(a)^{-1} \cdot \Phi(\delta)(a) = 1$$

for any $a \in G$. $\square$

By virtue of Lemma 3, the proof of Theorem is complete.

The Proof of Corollary. It is obvious that Lemmas 2 and 3 remain true, if in their formulations one changes $\mathrm{End}(G)$ to $\mathrm{Aut}(G)$. If δ is an automorphism, then $\Phi(\delta)$ also is an automorphism. Therefore, $\mathrm{Im}(\Phi(\delta)) = G$. Consequently, $\mathrm{Im}(\Phi(\delta))$ is not cyclic, because G is not a cyclic group. Using Lemma 3, we obtain $\Phi(\delta) = \delta$.

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