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Informatics

ASYMPTOTIC ESTIMATES OF THE NUMBER OF SOLUTIONS OF SYSTEMS OF EQUATIONS WITH DETERMINABLE PARTIAL BOOLEAN FUNCTIONS

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In this paper we investigate a class of equation systems with determinable partial (not everywhere defined) Boolean functions. We found the asymptotic estimate of the number of solutions of equation systems in the "typical" case (for the whole range of changes in the number of equations).

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Introduction. Many problems of discrete mathematics, including problems, which are traditionally considered to be complex, lead to the solutions of the systems of Boolean equations of the form

$$\begin{cases} f_i(x_1,\dots,x_n)=1, \\ i=1,\dots,l \end{cases} \tag{1}$$

or to the revealing of those conditions, under which the system (1) has a solution. In the general problem of realizing whether the system (l) has a solution or not is NP-complete [1]. Therefore, it is often necessary to consider special classes of the systems of equations, using their specificity, or explore a number of solutions in the "typical" case.

Some Necessary Definitions. Let $\{M(n)\}_{n=1}^{\infty}$ be a collection of sets such that $|M(n)|\underset{n\to\infty}{\longrightarrow}\infty$ $\big(|M|$ is the cardinality of the set $M\big)$, and $M^s(n)$ be the subset of all elements of $M(n)$, which have a property S. We say that almost all elements of the set $M(n)$ have a property S, if $\left|M^S(n)\right|/\left|M(n)\right|\underset{n\to\infty}{\longrightarrow}\infty$.

Lets denote by $S_{n,l}$ the set of all systems of the form (1), where $f_i(x_1,\dots,x_n)$, $i=1,\dots,l$, are pairwise different Boolean functions of variables $x_1,x_2,\dots,x_n$. It is easy to see that $|S_{n,l}|=C^l_{2^{2^n}}$.

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Let $B = \{0,1\}$, $B^n = \{\tilde{\alpha}/\tilde{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n), \alpha_i \in B, 1 \le i \le n\}$. The vector $\tilde{\alpha}_i = (\alpha_1, \alpha_2, \dots, \alpha_n) \in B^n$ is called a solution of (1), if

$$\begin{cases} f_i(\alpha_1, \alpha_2, \dots, \alpha_n) = 1, \\ i = 1, \dots, l. \end{cases}$$

We denote by $t(S)$ the number of solutions of the system S. It was found in [2, 3] asymptotics of the number of the solutions $t(S)$ for the almost all systems S from $S_{n,l}$ and it is done for the whole range of parameter l as $n \to \infty$. The paper [4] considers a special class of equation systems, where it is found the asyptotics behavior of the number of solutions.

In this work we consider a class of systems of equations with determinable partial (not everywhere defined) Boolean functions. We will give the asymptotic behavior of the number of solutions in a "typical" case.

A partial Boolean function at $\tilde{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in B^n$ is either not defined or takes values 0 or 1.. Let $Q(n)$ denote the set of all partial Boolean functions depending on variables $x_1, x_2, \dots, x_n$. Obviously, $|Q(n)| = 3^{2^n}$. Let $R(n,l)$ denote the set of systems of l equations of the form (1), where $f_i(x_1, \dots, x_n)$, $i = 1, \dots, l$, are pairwise different partial Boolean functions of variables $x_1, x_2, \dots, x_n$ ($f_i \ne f_j$, if $i \ne j$ condition holds). It is easy to see that $|R_{n,l}| = C^l_{3^{2^n}}$.

The vector $\tilde{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in B^n$ is called a solution of (1), if

$$\begin{cases} f_i(\alpha_1, \alpha_2, \dots, \alpha_n) \ne 0, \\ i = 1, \dots, l, \end{cases}$$

and at least for one of the functions $f_i(x_1, \dots, x_n)$, $i = 1, \dots, l$, we have $f_i(\alpha_1, \alpha_2, \dots, \alpha_n) = 1$. Namely, one can define a partial function not defined at $\tilde{\alpha}$ to by 1.

For the numbers of solutions $t(S)$ of almost all the systems S of $R(n,l)$ the following statement is true (here and further $f(n) \sim g(n)$, if $f(n)/g(n) \to 1$ as $n \to \infty$, $f(n) = o(g(n))$, if $f(n)/g(n) \to 0$ as $n \to \infty$. Everywhere below log stands for $\log_2$.

Theorem.

1. *If $n - \ell(\log 3 - 1) \to \infty$ as $n \to \infty$, then for almost all the systems S of $R(n,l)$ we have $t(S) \sim 2^n(2^l - 1)3^{-l}$.*

2. *If $n - \ell(\log 3 - 1) \to -\infty$ as $n \to \infty$, then almost all the systems S of $R(n,l)$ have no solutions.*

3. *If $n - \ell(\log 3 - 1)$ is restricted as $n \to \infty$, then for almost all the systems of $R(n,l,m)$ the number of solutions $t(S)$ has upper bound $\varphi(n)$, satisfying the condition $\varphi(n) \to \infty$ as $n \to \infty$.*

Proof. The following inequalities are known and easy to check:

a) Chebyshev's first inequality. Let a random variable $\xi \ge 0$ have mathematical expectation $M\xi$. Then for any $t > 0$ we have $P(\xi \ge t) \le M\xi/t$ [5].

b) Chebyshev's second inequality. Let a random variable ξ has a dispersion $D\xi$. Then for any $t > 0$ one has $P(|\xi - M\xi| \ge t) \le D\xi/t^2$ [5].

c) For any $x > 1$ we have $(1 - 1/x)^x < e^{-1}$.

d) For any natural n and $1 \le m \le n$ it holds $C_n^m < (en/m)^m$.

e) Let $b(k;n,p) = C_n^k p^k q^{n-k}$, where $0 < p,\ q < 1,\ p+q = 1$ and $r > np$. Then $\sum\limits_{j=0}^{n-r} b(r+j;n,p) < b(r;n,p)(r+1)q/(r+1-(n+1)p)$ (the estimate of the "tail" of the binomial distribution [5]).

Let S be a system from $R(n,l)$. Arranging the equations in S, we obtain $l!$ systems of equations. Thus, from the set $R(n,l)$ we obtain a new set $R'(n,l)$ of ordered systems. It's evident that

$$|R'(n,l)| = |R(n,l)|l!. \tag{2}$$

Suppose that almost all the systems of $R'(n,l)$ have a property E, which is invariant against the rearrangements of the equations. It's easy to see that almost all the systems of $R(n,l)$will also have the property E. Thus to proof of the Theorem it will be enough to consider the set $R'(n,l)$ instead of $R(n,l)$. Next, we denote by $R''(n,l)$ the extension of $R'(n,l)$, where the systems from $R'(n,l)$ can have the same equations. It is easy to see that

$$|R"(n,l)| = 3^{l2^n}. \tag{3}$$

From (2), (3) and d) we obtain

$$\frac{|R'(n,l)|}{|R''(n,l)|} = \frac{l!C^l_{3^{2^n}}}{3^{l2^n}} \to 1,$$

when $l^2 = o\left(3^{2^n}\right)\ (n \to \infty)$. Thus, if $l^2 = o\left(3^{2^n}\right)$, then any assertion for the almost all systems of $R''(n,l)$ is true also for almost all the systems of $R'(n,l)$.

We consider $R''(n,l)$ as a space of events, where every event $S \in R''(n,l)$ holds with the probability $1/|R''(n,l)| = 3^{-l2^n}$. Consider the random value $\xi_S(\tilde{\alpha})$, which is connected with $S \in R'(n,l)$ as follows:

$\xi_S(\tilde{\alpha}) = 1$, if $\tilde{\alpha}$ is the solution of the system S, and $\xi_S(\tilde{\alpha}) = 0$ otherwise.

From the definition it follows that the number of the system $S \in R'(n,l)$, for which $\tilde{\alpha}$ is a solution, is equal to $(2^l-1)3^{l(2^n-1)}$. From this and (3) it follows that $P(\xi_S(\tilde{\alpha}) = 1) = (2^l-1)3^{-l}$. Let $p(l) = P(\xi_S(\tilde{\alpha}) = 1) = (2^l-1)3^{-l}$, $q(l) = P(\xi_S(\tilde{\alpha}) = 0) = 1 - p(l) = 1-(2^l-1)3^{-l}$. For the mathematical expectation $M\xi_S(\tilde{\alpha})$ and dispersions $D\xi_S(\tilde{\alpha})$ of the random variable $\xi_S(\tilde{\alpha})$ we get

$$M\xi_S(\tilde{\alpha}) = p(l) = (2^l-1)3^{-l},\ D\xi_S(\tilde{\alpha}) = p(l) - p^2(l) =$$
$$= p(l)q(l) = (2^l-1)3^{-l}(1-(2^l-1)3^{-l}).$$

Consider another random value $\nu = \sum\limits_{\tilde{\alpha}\in B^n} \xi_S(\tilde{\alpha})$, which is the number of solutions of the system S. Random value ν has a binomial distribution, because

$$p(\nu = j) = C^j_{2^n} 3^{-lj}(1-3^{-l})^{2^n-j}.$$

Hence, $M\nu = 2^n 3^{-l}$and $D\nu = 2^n 3^{-l}\left(1-3^{-l}\right)$, where $M\nu$ and $D\nu$ are the mathematical expectation and dispersion of the random value ν respectively. In fact,

$$M\nu = \sum_{\alpha\in B^n} M\xi_S(\tilde{\alpha}) = 2^n p(l) = 2^n(2^l-1)3^{-l},$$

$$D\nu = \sum_{\alpha\in B^n} D\xi_S(\tilde{\alpha}) = 2^n p(l)q(l) = 2^n(2^l-1)3^{-l}\left(1-(2^l-1)3^{-l}\right).$$

Let $n-\ell(\log 3-1) \to \infty$ as $n \to \infty$. It means that $Mv = 2^n(2^l-1)3^{-l} = 2^{n-l(\log 3-1)}(1-2^{-l}) \to \infty$ as $n \to \infty$. Using the Chebishev's inequality b) for $t = Mv/\sqrt{n-l(\log 3-1)}$, we obtain

$$P\frac{(|v-Mv| \geq Mv}{\sqrt{n-l(\log 3-1)}} \leq \frac{(n-l(\log 3-1))(1-(2^l-1)3^{-l})}{(2^{n-l(\log 3-1)}(1-2^{-l}))} \underset{n\to\infty}{\longrightarrow} 0.$$

Hence, from the definition of random value v it follows that almost all the systems of $R''(n,l)$ have number of solutions which asymptotically equals to Mv. Since $n-\ell\log 3 \to \infty$ and $l^2 = o\left(3^{2^n}\right)$, almost all the systems of $R(n,l)$ have also number of solutions, asymptotically equal to $Mv = 2^n(2^l-1)3^{-l}$. The first statement of the Theorem is proved.

Let $n-\ell(\log 3-1) \underset{n\to\infty}{\longrightarrow} -\infty$. Then

$$Mv = 2^n(2^l-1)3^{-l} = 2^{n-l(\log 3-1)}(1-2^{-l}) \to 0 \ \ (n \to \infty).$$

Using Chebishev's first inequation when $t = l$, we obtain $P(v \geq 1) \to 0$ as $n \to \infty$ and, therefore, $P(v = 0) \to 1$ as $n \to \infty$. Hence, it follows that almost all the systems S of $R''(n,l)$ have no solution. Therefore, $l^2 = o\left(3^{2^n}\right)$ the second statement of the Theorem is proved. It is easy to see that for greater values of the parameter l the statement of the Theorem also holds (the number of solutions of the system does not increase as the number of equation increases).

Now let $n-\ell(\log 3-1)$ is bounded as $n \to \infty$. Then $Mv = 2^n(2^l-1)3^{-l} = 2^{n-l(\log 3-1)}(1-2^{-l})$ is also bounded. Using the inequations 5), 4) and 3), we obtain

$$P(v > r) = \sum_{i=0}^{2^n-r} C_{2^n}^{r+i}((2^l-1)3^{-l})^{r+i}(1-(2^l-1)3^{-l})^{2^n-r-i} \leq$$

$$\leq C_{2^n}^{r}\frac{(3^{-l}(2^l-1))^r(1-3^{-l}(2^l-1))^{2^n-r}(r+1)(1-3^{-l}(2^l-1))}{r+1-3^{-l}(2^n+1)(2^l-1)} \leq$$

$$\leq (e2^n3^{-l}r^{-1}(2^l-1))^r \leq (eMv/r)^r) \underset{r\to\infty}{\longrightarrow} 0,$$

because Mv is bounded. Putting $r = \varphi(n)$, where $\varphi(n)$ is an arbitrary function $\varphi(n)$, satisfying the condition $\varphi(n) \to \infty$ as $n \to \infty$, we obtain $P(v \leq \varphi(n)) \to 1$ when $n \to \infty$. Therefore, for almost all the systems of $R''(n,l)$ the third statement of the Theorem holds. Since $n-\ell(\log 3-1)$ is bounded, we get $l^2 = o\left(3^{2^n}\right)$ and therefore for almost all the systems of $R(n,l)$ it holds the third statement of the Theorem.

Theorem is completely proved.

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Է. Վ. ԵՂԻԱԶԱՐՅԱՆ

ՈՐՈՇԵԼԻ ՄԱՍՆԱԿԻ ԲՈՒԼՅԱՆ ՖՈՒՆԿՑԻԱՆԵՐՈՎ ՀԱՎԱՍԱՐՈՒՄՆԵՐԻ ՀԱՄԱԿԱՐԳԵՐԻ ԼՈՒԾՈՒՄՆԵՐԻ ՔԱՆԱԿԻ ՍԱՀՄԱՆԱՅԻՆ ԳՆԱՀԱՏԱԿԱՆՆԵՐ

Այս աշխապանքում հեփազոփվում են որոշելի մասնակի (ոչ ամենուրեք որոշված) բուլյան ֆունկցիաներից կազմված հավասարումների համակարգեր: Տրվում են հավասարումների համակարգերի լուծումների այդ քանակի սահմանային գնահապականներ “փիպիկ” դեպքում (հավասարումների քանակի փոփոխման ամբողջ փիրույթի համար):