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M a t h e m a t i c s

ON THE POSSIBILITY OF GROUP-THEORETIC DESCRIPTION OF AN EQUIVALENCE RELATION CONNECTED TO THE PROBLEM OF COVERING SUBSETS IN FINITE FIELDS WITH COSETS OF LINEAR SUBSPACES

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Let F_q^n be an n-dimensional vector space over a finite field F_q. Let $C(F_q^n)$ denote the set of all cosets of linear subspaces in F_q^n. Cosets $H_1, H_2, \ldots, H_s$ are called exclusive if $H_i \not\subseteq H_j$, $1 \leq i < j \leq s$. A permutation f of $C(F_q^n)$ is called a C-permutation, if for any exclusive cosets $H, H_1, H_2, \ldots, H_s$ such that $H \subseteq H_1 \cup H_2 \cup \cdots \cup H_s$ we have: *i)* cosets $f(H), f(H_1), f(H_2), \ldots, f(H_s)$ are exclusive; *ii)* cosets $f^{-1}(H), f^{-1}(H_1), f^{-1}(H_2), \ldots, f^{-1}(H_s)$ are exclusive; *iii)* $f(H) \subseteq f(H_1) \cup f(H_2) \cup \cdots \cup f(H_s)$; *vi)* $f^{-1}(H) \subseteq f^{-1}(H_1) \cup f^{-1}(H_2) \cup \cdots \cup f^{-1}(H_s)$.

In this paper we show that the set of all C-permutations of $C(F_q^n)$ is the General Semiaffine Group of degree n over F_q.

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Introduction. Shannon and Povarov introduced an equivalence relation on the set of Boolean functions in relation to Boolean function synthesis by switching circuits [1, 2]. Two Boolean functions of n variables are called equivalent if they can be transformed into each other by an isometric transformation of the vertices of the n-dimensional unit cube E^n. Isometric transformations form a group (Shannon–Povarov group) generated by permutations of the variables and negations of some of the variables.

It is easy to verify that for equivalent Boolean functions the complexities of synthesis by Disjunctive Normal Forms (DNF) and by switching circuits are equal. Tabulating of Shannon–Povarov classes reduces the problem of optimal synthesis of a given Boolean function in the class of DNF or switching circuits to finding an equivalent representative in the Table.

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Due to high number of equivalence classes Shannon–Povarov tabulating is practically not solvable even for $n = 5$. In [3] a new equivalence relation is considered in a hope to make the tabulating problem easier.

Let f be a Boolean function of n arguments and $N_f \subseteq E^n$ be the subset of points on which f is 1. A subset of points $N \subseteq E^n$ corresponding to a conjunction K is called an interval. An interval $N_1 \subseteq N_f$ is called a maximal interval for f, if there is no interval $N_2 \subseteq N_f$ such that $N_1 \subset N_2$. A DNF $K_1 \vee K_2 \vee \cdots \vee K_s$ of the function f, which corresponds to a covering of the set N_f by all the maximal intervals of f is called the reduced DNF of function f. The set of all maximal intervals of f is denoted by D_f. Functions f and g are called equivalent, if there is a bijection $h : D_f \to D_g$ such that the condition $N_1 \subseteq M_1 \cup M_2 \cup \cdots \cup M_s, N_1, M_i \in D_f, 1 \leq i \leq s$ holds if and only if $h(N_1) \subseteq h(M_1) \cup h(M_2) \cup \cdots \cup h(M_s), h(N_1), h(M_i) \in D_g, 1 \leq i \leq s$. A covering of N_f by a subset of D_f is called an irreducible covering, if it ceases to be a covering upon removal of any of its intervals. A DNF corresponding to an irreducible covering is called a terminal DNF. The length of a DNF is the number of its intervals. The shortest DNF of f is a DNF of f with the least possible length. Clearly for equivalent functions f and g the image of any terminal DNF in f is a terminal DNF in g and vice versa, and the lengths of their shortest DNFs are equal. The group of isometric transformations of E^n acts naturally on the set of all intervals of E^n and functions that are in the same orbit are equivalent to each other. It is shown in [3], that there is no larger group with this property, i.e. every bijection on the set of all intervals with this property is an isometric transformation.

The problem of finding of the shortest coset covering was introduced in [4] originally for Boolean functions in relation with a natural generalization of the notion of DNF of Boolean functions. Let F_q stand for a finite field with q elements [5], and F_q^n, $n \geq 2$, for an n-dimensional linear space over F_q (obviously F_q^n is isomorphic to F_{q^n}). If L is a linear subspace in F_q^n, then the set $\alpha + L \equiv \{\alpha + x \mid x \in L\}, \alpha \in F_q^n$, is a coset (or translate) of the subspace L and $\dim(\alpha + L)$ coincides with $\dim(L)$. An equivalent definition: a subset $N \subseteq F_q^n$ is a coset if whenever $x^1, x^2, \ldots, x^m$ are in N, so is any affine combination of them, i.e. so is $\sum_{i=1}^{m} \lambda_i x^i$ for any $\lambda_1, \lambda_2, \ldots \lambda_m$ in F_q such that $\sum_{i=1}^{m} \lambda_i = 1$. The set of all cosets in F_q^n is denoted by $C(F_q^n) (F_q^n \notin C(F_q^n))$. A k-dimensional coset is called a k-coset. It can be readily verified that any k-coset in F_q^n can be represented as a set of solutions of a certain system of linear equations over F_q of rank $n-k$ and vice versa.

Definition 1. A set M of cosets forms a coset covering for a subset N in F_q^n if and only if $N = \bigcup_{H \in M} H$. The number of cosets in M is the length (or complexity) of the covering. The shortest coset covering is the covering of the minimal possible length.

The subset $N \subseteq F_q^n$ can be given in different ways: as a list of elements, as a set of solutions of a polynomial equation over F_q^n etc. Finding the shortest coset covering means finding the minimal number of systems of linear equations over F_q such that

N coincides with the union of solutions of the linear systems. Various aspects of this problem were investigated in [6–11]. In this paper we consider the analogue of the problem considered in [3] with a more general condition.

Definition 2. Cosets $H_1, H_2, \ldots, H_s$ are called exclusive, if $H_i \not\subseteq H_j$, $1 \le i < j \le s$.

Definition 3. A permutation f of $C(F_q^n)$ is called a C-permutation, if for any exclusive cosets $H, H_1, \ldots, H_s$ such that $H \subseteq H_1 \cup H_2 \cup \cdots \cup H_s$, we have

i) cosets $f(H), f(H_1), f(H_2), \ldots, f(H_s)$ are exclusive;

ii) cosets $f^{-1}(H), f^{-1}(H_1), f^{-1}(H_2), \ldots, f^{-1}(H_s)$ are exclusive;

iii) $f(H) \subseteq f(H_1) \cup f(H_2) \cup \cdots \cup f(H_s)$;

iv) $f^{-1}(H) \subseteq f^{-1}(H_1) \cup f^{-1}(H_2) \cup \cdots \cup f^{-1}(H_s)$.

Let f be a C-permutation and $H_1, H_2, \ldots, H_s$ be a list of exclusive cosets in F_q^n. If $1 \le i_1 < i_2 < \cdots < i_k \le s$, then $H_{i_1} \cup H_{i_2} \cup \cdots \cup H_{i_k} = H_1 \cup H_2 \cup \cdots \cup H_s$ if and only if $f(H_{i_1}) \cup f(H_{i_2}) \cup \cdots \cup f(H_{i_k}) = f(H_1) \cup f(H_2) \cup \cdots \cup f(H_s)$.

Definition 4. A permutation f of F_q^n is called semiaffine, if there is an automorphism σ of F_q, a permutation g of F_q^n and a vector $b \in F_q^n$ such that for all x, y in F_q^n and λ in F_q it holds that

i) $g(x+y) = g(x) + g(y)$;

ii) $g(\lambda x) = \sigma(\lambda) g(x)$;

iii) $f(x) = g(x) + b$.

If $q = p^m$, p is prime, then $\sigma^0, \sigma^1, \ldots, \sigma^{m-1}$, where $\sigma^k : x \to x^{p^k}$, are all the automorphisms of F_q (Theorem 2.21, [5]). For $q = p$ the only automorphism is the identity.

Definition 5. If the automorphism σ in the previous definition is the identity, then f is said to be affine.

The general semiaffine (affine) group of degree n over F_q, denoted by $\Gamma A(n, F_q)$ ($Aff(n, F_q)$), is the group of all semiaffine (affine) permutations of F_q^n. If q is a prime, then $\Gamma A(n, F_q) = Aff(n, F_q)$. Two groups act naturally on $C(F_q^n)$ and coset dimension remains invariant under this action. Therefore, a semiaffine permutation of F_q^n can also be considered as a permutation of $C(F_q^n)$. Clearly every semiaffine transformation is a C-permutation. In this article we consider the problem whether there is another group that acts on $C(F_q^n)$ and satisfies this property. The following theorem gives an answer to that question.

Theorem . A permutation f of $C(F_q^n)$ is a C-permutation if and only if f is semiaffine.

Proof of the Theorem. As the semiaffine condition in the theorem is clearly sufficient for a permutation to be a C-permutation, it is only left to prove the necessity.

Lemma 1. Intersection of two cosets in F_q^n is either empty or is a coset of the intersection of their corresponding linear subspaces.

Proof. Let L_1 and L_2 be linear subspaces in F_q^n and $x, y \in F_q^n$. Suppose $(x+L_1) \cap (y+L_2)$ is not empty and $z \in (x+L_1) \cap (y+L_2)$. Then $x + L_1 = z + L_1$ and $y + L_2 = z + L_2$. It is easy to see that $z + (L_1 \cap L_2) \subseteq (z+L_1) \cap (z+L_2)$. Now if $z + l_1 = z + l_2$ for some $l_1 \in L_1, l_2 \in L_2$, then $l_1 = l_2 \in L_1 \cap L_2$, and $(z+L_1) \cap$

$(z+L_2) \subseteq z+(L_1 \cap L_2)$. Thus, $(z+L_1) \cap (z+L_2) = z+(L_1 \cap L_2)$, which completes the Proof. □

By $span(S)$ we denote the linear span of the set S.

L e m m a 2. Let $H_1 \subseteq F_q^n$ be a k-coset, $1 \le k \le n-1$. Then:

$i)$ there exists a 1-coset $H_2 \subseteq F_q^n$ such that $\dim(H_1 \cap H_2) = 0$;

$ii)$ there exists a 1-coset $H_3 \subseteq F_q^n, H_3 \neq H_2$ such that $H_3 \cap H_1 =$

$= H_3 \cap H_2 = H_1 \cap H_2$.

P r o o f.

$i)$ Let L_1 be the linear subspace of H_1 and $H_1 = x+L_1$ for some $x \in F_q^n$. Let $a_1,a_2,\dots,a_k$ be the basis of L_1. Then the vectors $a_1,a_2,\dots,a_k,b$, where $b \in F_q^n \backslash L_1$ are linearly independent. Set $L_2 = span(\{b\})$. Suppose $c \in L_1 \cap L_2$, i.e. $c = \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_k a_k = \beta b$ for some $\alpha_i, \beta \in F_q, 1 \le i \le k$. Then $0 = \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_k a_k - \beta b$, which implies $\alpha_i = \beta = 0, 1 \le i \le k$, and $c = 0$. Hence $L_1 \cap L_2 = \{0\}$. Taking $H_2 = x+L_2$, implies $\dim(H_1 \cap H_2) = 0$ as claimed.

$ii)$ Let $c \in F_q^n \backslash (L_1 \cup L_2)$ and $H_3 = x+span(\{c\})$.

Clearly, the second assertion holds. □

L e m m a 3. Let f be a C-permutation. Then:

$i)$ f takes k-cosets to k-cosets, $0 \le k \le n-1$;

$ii)$ if $H_1 = \{h_1,h_2,\dots,h_s\}$ is a k-coset,$0 \le k \le n-1$, then so is

$H_2 = \{f(h_1), f(h_2),\dots, f(h_s)\}$ and $f(H_1) = H_2$.

P r o o f.

$i)$ Suppose to the contrary there exist cosets H_1, H_2 such that $f(H_1) = H_2$ and without loss of generality we may assume that $k_1 \equiv \dim(H_1) > \dim(H_2) \equiv k_2$. From Lemma 2 it follows that there is a 1-coset M_1 such that $\dim(H_1 \cap M_1) = 0$. Let L be the corresponding linear subspace of M_1. Suppose $H_1 = \{h_1,h_2,\dots h_s\}$, $s = q^{k_1}$, and $M_1 = h_1 + L$. Set $M_i = h_i + L$, $2 \le i \le s$. From Lemma 1 it follows that $\dim(H_1 \cap M_i) = 0$, $2 \le i \le s$. Then cosets $H_1, M_1, M_2, \dots, M_s$ are exclusive and $H_1 \subseteq M_1 \cup M_2 \cup \dots \cup M_s$. Hence, $H_2 \subseteq f(M_1) \cup f(M_2) \cup \dots \cup f(M_s)$. The exclusivity of $H_2, f(M_1), f(M_2), \dots, f(M_s)$ implies $k_2 \neq 0$. If $H_2 \subseteq (f(M_1) \cup f(M_2) \cup \dots \cup f(M_s)) \backslash f(M_i)$, for some $1 \le i \le s$, then $H_1 \subseteq (M_1 \cup M_2 \cup \dots \cup M_s) \backslash M_i$, which contradicts $M_i \cap M_j = \emptyset$. Thus, there exist $f_i \in f(M_i) \cap H_2, 1 \le i \le s$, and $f_i \notin f(M_j), i \neq j$. Now $\{f_1, f_2, \dots, f_s,\} \subseteq H_2$ and $|H_2| \ge s$, which contradicts $\dim(H_1) > \dim(H_2)$.

$ii)$ Again assume to the contrary $x = f^{-1}(h) \notin H_1$ for some $h \in H_2$. If $f^{-1}(h) \notin M_i, 1 \le i \le s$, then $f^{-1}(h), H_1, M_1, M_2, \dots, M_s$ are exclusive, $H_1 \subseteq f^{-1}(h) \cup M_1 \cup M_2 \cup \dots \cup M_s$, but $h, H_2, f(M_1), f(M_2), \dots, f(M_s)$ is not exclusive and we have a contradiction. If $f^{-1}(h) \in M_i$ for some $1 \le i \le s$, then by Lemma 2 we can replace M_i with a 1-coset $\tilde{M}_i$, so that cosets $H_1, \{M_1, M_2 \dots, M_s\} \backslash M_i, \tilde{M}_i$ are exclusive,$H_1 \subseteq \bigcup_{j \neq i} M_j \cup \tilde{M}_i$, and $f^{-1}(h) \notin \bigcup_{j \neq i} M_j \cup \tilde{M}_i$. The case is now reduced to the case that we just covered. □

It is well known, that if a permutation f of F_q^n, where $q \neq 2$, maps 1-cosets to 1-cosets, then f is semiaffine. In the case of $q = 2$ a 1-coset in F_2^n is just a two element subset, hence every permutation of F_2^n takes all 1-cosets to 1-cosets. However, a permutation of F_2^n, which takes every 2-coset to a 2-coset, must be affine [12]. Now the necessity of the Theorem is immediate from Lemma 3.

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