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Mathematics

ON ALGEBRAIC EQUATION WITH COEFFICIENTS FROM THE β-UNIFORM ALGEBRA $C_\beta(\Omega)$

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In this work the question of algebraic closeness of β-uniform algebra $A(\Omega)$ defined on locally compact space Ω is investigated.

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Introduction. In the present work algebraic equations of the following type

$$\lambda^n + a_1(x)\lambda^{n-1} + \cdots + a_n(x) = 0 \qquad (*)$$

are investigated, where $a_j(x)$, $j = 1, \ldots, n$, are a complex valued, bounded and continuous functions given on some locally compact Hausdorff space Ω. The aim of this work is to obtain the conditions, which provide solvability of equation $(*)$ in the algebra of complex-valued, boundary and continuous functions on the space Ω. If instead of an individual equation $(*)$, we consider a class of equations $(*)$, then the question about the description of a locally compact space Ω, on which any equation of type $(*)$ are solvable, became interesting.

We note that for the compacts this problem were sufficiently detailed studied in the works [1–3].

Let Ω be a locally compact Hausdorff space. We assume that the space Ω admits a "compact exhaustion", that is there exists a compacts $K_p \subset \Omega$ such that $K_p \subset K_{p+1}$ and $\Omega = \bigcup\limits_{p=1}^{\infty} K_p$. As such locally compact it can be considered the following set $\Omega = \left\{(x,y) \in \mathbb{R}^2 : y = \sin\frac{1}{x},\ 0 < x < \frac{2}{\pi}\right\}$ and as K_p the set $K_p = \left\{(x,y) \in \Omega : \frac{1}{p} \leqslant x \leq \frac{2}{\pi} - \frac{1}{p}\right\}$, where $p = 4, 5, \ldots$

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Simultaneously, we note that a locally compact Ω is called a "hereditarily unicoherent", if for any two connected closed subsets $K_1, K_2 \subset \Omega$ their intersection is also a connected set.

Since the studied below algebras are topological algebras, more precisely a β-uniform algebras, we give their description.

Let $C_\infty(\Omega)$ be an algebra of all complex-valued, bounded and continuous functions given on a locally compact Hausdorff space Ω. Then this algebra allotted with uniform norm $\left(\text{i.e. } \|f\|_\infty = \sup\limits_{\Omega}|f(x)|\right)$ becomes a Banach algebra, which we denote by $C_b(\Omega)$. At the same time, using the ideal $C_0(\Omega) \subset C_\infty(\Omega)$ of functions vanishing at infinity (i.e. for each $f \in C_0(\Omega)$ and $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset \Omega$ such that $|f||_{\Omega\setminus K_\varepsilon} < \varepsilon$) one can introduce a topology on the algebra $C_\infty(\Omega)$ by the family of algebraic seminorms $\{P_g\}_{g\in C_0(\Omega)}$, where $P_g(f) = \|T_g f\|_\infty$ and $T_g : C_b(\Omega) \to C_b(\Omega)$ is the multiplication operator $T_g f = gf$.

Natural topology on $C_\infty(\Omega)$ given by this family of algebraic seminorms is called a β-uniform topology and the algebra $C_\infty(\Omega)$ in this topology will be denoted by $C_\beta(\Omega)$ following the notation of [4–6]. Thus, a β-uniform topology and the algebra $C_b(\Omega)$ is the weakest of the topologies, under which all linear operators $\{T_g\}_{g\in C_0(\Omega)} \subset BL(C_b(\Omega))$, i.e. the base of neighborhood of zero is given by the sets

$$U(g_1,\dots,g_n;\varepsilon) = \left\{f \in C_b(\Omega) : P_{g_i}(f) = \|T_{g_i} f\|_\infty < \varepsilon, \text{ where } g_i \in C_0(\Omega);\ i = 1,\dots,n\right\}.$$

We recall that a subalgebra $\mathcal{A}(\Omega)$ of the algebra $C_\beta(\Omega)$ is called β-uniform algebra on Ω, if it is a closed subalgebra of a β-uniform algebra $C_\beta(\Omega)$, contains constants and separates the points of Ω.

We note that an interesting difference between the uniform and the β-uniform algebras is observed by the fact that on a uniform algebra every complex homomorphism is continuous, as for the β-uniform algebras this is not true. For example, on a uniform algebra $C_b(\Omega)$ every complex homomorphism is continuous, since $C_b(\Omega)^* = \mathcal{M}(b\Omega)$, where $b\Omega$ is a Stone-Cech compactification of Ω, which is the of maximal ideals space of the uniform algebra $C_b(\Omega)$. On the other hand, for the β-uniform of the algebra $C_\beta(\Omega)$ we have $C_\beta(\Omega)^* = \mathcal{M}(\Omega)$ (see [6, 7]), where $\mathcal{M}(\Omega)$ is the space of all bounded complex regular measures on Ω, then all the pointing functionals corresponding to the points of $b\Omega \setminus \Omega$ are discontinuous complex homomorphisms on the β-uniform algebra $C_\beta(\Omega)$. It is interesting to note that, in the context of the foregoing, it is not known whether there exists a Frechet algebra on which there is a discontinuous complex homomorphism.

From the above definition of the base of neighbourhood of zero in a β-uniform topology it follows that the family of seminorms $\{P_F\}_{F\in\mathcal{F}(\Omega)}$, where $F = \{g_{i_1},\dots,g_{i_n}\}$ runs troughs the set of all finite subsets $\mathcal{F}(\Omega)$ from $C_0(\Omega)$ and $P_F(f) = \sum\limits_{g\in F}\|T_g f\|_\infty$ is a directed family of seminorms, which defines a β-uniform topology on the algebra $C_\beta(\Omega)$.

This allows us to represent a β-uniform algebra $\mathcal{A}(\Omega)$ as a projective limit of the system of uniform algebras $(\mathcal{A}_F(\Omega);\pi_{F,H})$, i.e. $\mathcal{A}(\Omega) = \varprojlim(\mathcal{A}_F(\Omega);\pi_{F,H})$

(see [7, 8]). Then the set $M_{\mathcal{A}(\Omega)}$ of all β-uniform continuous linear multiplicative functionals are inductive limits of maximal ideals of the space of uniform algebra $\mathcal{A}_F(\Omega)$.

Theorem 1. Let Ω be a locally connected, locally compact Hausdorff space, which admits a compact exhaustion, and $\mathcal{A}(\Omega)$ is a β-uniform algebra on a Ω such that for each $f \in \mathcal{A}(\Omega)$ there exists a natural number $k = k(f) \geqslant 2$ and $g \in \mathcal{A}(\Omega)$ such that $g^k = f$. Then $\mathcal{A}(\Omega) = C_\beta(\Omega)$.

Proof. Since a locally compact Ω is a locally connected and admits a "compact exhaustion", we have $\Omega = \bigcup_{p=1}^{\infty} K_p$, where $K_p \subset \Omega$ is a locally connected compacts. Let $\mathcal{A}(K_p)$ is a uniform algebras on K_p such that $\mathcal{A}(\Omega) = \varprojlim (\mathcal{A}(K_p); \pi_{p,q})$. Since a β-uniform algebra $\mathcal{A}(\Omega)$ by condition is binomial solvable, for each natural p a uniform algebra $\mathcal{A}(K_p)$ is binomial solvable too. Then by the Theorems from [2,3], we have $\mathcal{A}(K_p) = C(K_p)$, from which it follows that $\mathcal{A}(\Omega) = \varprojlim (\mathcal{A}(K_p); \pi_{p,q}) = \varprojlim (C(K_p); \pi_{p,q}) = C_\beta(\Omega)$. □

The statement below is an analogous of R. Countryman's Theorem (see [1]) for a β-uniform algebras.

Theorem 2. Let Ω be a connected, locally compact Hausdorff space that admits a compact exhaustion. Then a β-uniform algebra $C_\beta(\Omega)$ will be algebraically closed if and only if the space Ω is a locally connected and hereditarily unicoherent.

The proof follows from the fact that $C_\beta(\Omega) = \varprojlim (C(K_p); \pi_{p,q})$ and each algebra $C(K_p)$ is algebraically connected (see mentioned above R. Countryman's Theorem).

We consider now the class of equations for which for each $x_0 \in \Omega$ the corresponding equation with numerical coefficients does not have a multiple roots. We are interested in a condition on Ω guaranteeing a solvability of this equations.

We define the class of all equations $(*)$ without multiple roots by $\mathfrak{A}_n(\Omega)$ (see [9]) and $\overline{\mathfrak{A}_n(\Omega)} = \bigcup_{k \leqslant n} \mathfrak{A}_k(\Omega)$.

The set $\mathfrak{A}_n(\Omega)$ turns into a metric space with respect the metric $\rho(f, \tilde{f}) = \sup_{x \in \Omega} \left(\sqrt{\sum_{j=1}^{n} \left| a_j(x) - \tilde{a}_j(x) \right|^2} \right)$, where a_j, $\tilde{a}_j$ are the corresponding coefficients of the equations $f, \tilde{f} \in \mathfrak{A}_n(\Omega)$.

Simultaneously we note that for a connected, finite latticed complex Ω (see [9, 10]) the question about solvability on Ω an algebraic equations of type $(*)$ is connected with the fundamental group $\pi_1(\Omega)$, namely the group $H^1(\Omega, \mathbb{Z})$ that is isomorphic to the group $Hom(\pi_1(\Omega); \mathbb{Z})$. It is shown in [9], that for a connected finite latticed complex Ω missing of a nontrivial homomorphism of a group $\pi_1(\Omega)$ into a Artin's group of a "braid" $\mathcal{B}_n$ is equivalent to the fact that any equation of type $(*)$ without multiple roots is completely solvable, i.e. they belong to the class $\overline{\mathfrak{A}_n(\Omega)}$.

T h e o r e m 3. Let Ω be a connected, locally compact Hausdorff space, which admits a connected compact exhaustion (i.e. $\Omega = \bigcup\limits_{p=1}^{\infty} K_p$, where K_p are a connect compacts). Suppose that for each K_p there exists a sequence of inverse spectrum of a connected, finite latticed complexes $(K_{p,\alpha};\omega_\alpha)$ converging to K_p, such that all $\pi_1(K_{p,\alpha};\omega_\alpha)$ are commutative groups. Then necessary and sufficient condition for a complete solvability of all equations from the class $\overline{\mathfrak{A}_n(\Omega)}$ is the condition that the group $H^1(\Omega;\mathbb{Z})$ is divisible by $n!$.

P r o o f . Note that (see [8]) according to

$$H^1(\Omega;\mathbb{Z}) = C_\beta^{-1}(\Omega)/\exp(C_\beta)(\Omega) = \varinjlim C^{-1}(K_p)/\exp(C(K_p))$$

and the Arens–Roiden's Theorem (see [11]), we have $C^{-1}(K_p)/\exp(C(K_p)) = = H^1(K_p;\mathbb{Z})$ such that $H^1(\Omega,\mathbb{Z}) = \varinjlim H^1(K_p;\mathbb{Z})$. On the other hand, for each $p \in \mathbb{N}$ we have $H^1(K_p;\mathbb{Z}) = \varinjlim H^1(K_{p,\alpha};\mathbb{Z})$, where $\varinjlim$ denotes the inductive limit. As shown in [9], if K_p is a connected compact such that there exists an inverse spectrum of connected, finite latticed complexes, which converge to K_p, where all groups $\pi_1(K_{p,\alpha})$ are commutative, then a complete solvability of all equations from $\overline{\mathfrak{A}_n(K_p)}$ is equivalent to the divesibility of the group $H^1(K_{p,\alpha};\mathbb{Z})$ by $n!$.

Since

$$H^1(\Omega;\mathbb{Z}) = \varinjlim H^1(K_p;\mathbb{Z}) \quad \text{and} \quad H^1(K_p;\mathbb{Z}) = \varinjlim H^1(K_{p,\alpha};\mathbb{Z})$$

we complete the proof of Theorem. □

We will derive below an interesting application of the above results in the algebra of $\mathcal{L}$-convolution operators arising in the self-adjoint differential operator on $L^2(\mathbb{R})$.

Let $\mathcal{L}$ be a self-adjoint operator on $L^2(\mathbb{R})$ generated by the differential expression

$$(\ell y)(x) = -y''(x) + q(x)y(x)$$

with a real potential $q(x)$ satisfying the condition $(1+|x|)q(x) \in L^1(\mathbb{R})$, and let $u^{\pm}(x,\lambda)$ $(x,\lambda \in \mathbb{R})$ be the solutions of the equation $\ell y = \lambda^2 y$ that are eigenfunctions of the right and left scattering problems, respectively, which represent a complete orthonormal set of eigenfunctions of the continuous spectrum (see [12, 13]).

The operators $\tau, m(a), I : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, where $a \in C_\beta(\mathbb{R})$, and $\tau(y,(x)) = y(-x)$, $m(a)y = ay$, $Iy = y$, generate the operators $U_\pm, U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, where $(U_\pm y)(\lambda) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} u^{\pm}(\lambda,x)y(x)\,dx$, $\lambda \in \mathbb{R}$, $U = m(\chi_+)U_- + m(\chi_-)U_+$, $\chi_\pm$ is the characteristic function corresponding to the set $\mathbb{R}_\pm$ and the integrals are understood in the sense of convergence in $L^2(\mathbb{R})$.

Then the operators $U_\pm$ are bounded operators, the operator U is a partial isometry and $U^*U = I - P$, $UU^* = P$, where P is the projection of $L^2(\mathbb{R})$ onto a proper subspace corresponding to the discrete spectrum (see [12]).

Let $\mathcal{A}(\mathbb{R})$ be a β-uniform subalgebra in the algebra $C_\beta(\mathbb{R})$. Denote by $\mathcal{A}_{\mathcal{L}}(\mathbb{R})$ the algebra of $\mathcal{L}$-convolution operators of the form $U^*m(a)U$ on $L^2(\mathbb{R})$. We note that $\mathcal{A}_{\mathcal{L}}(\mathbb{R}) = \mathcal{A}(\mathbb{R})$ up to isomorphism and the following corollary holds.

Corollary 1. If the algebra of $\mathcal{L}$-convolution operators $\mathcal{A}_{\mathcal{L}}(\mathbb{R})$ is binomially solvable, then $\mathcal{A}_{\mathcal{L}}(\mathbb{R}) = C_{\beta}(\mathbb{R})$.

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