ԵՐԵՎԱՆԻ ՊԵՏԱԿԱՆ ՀԱՄԱԼՍԱՐԱՆ

Տիգրան Լևոնի Հակոբյան

Ձևական մոդուլների սիմպլեկտիկ տարածություն

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Symplectic space of formal modules

SYNOPSIS

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Ատենախոսության թեման հաստատվել է ԵՊՀ մաթեմատիկայի և մեխանիկայի ֆակուլտետի խորհրդի կողմից:

Գիտական ղեկավար՝	ֆիզ-մաթ. գիտ. դոկտոր Վ. Աթաբեկյան
Պաշտոնական ընդդիմախոսներ՝	ֆիզ-մաթ. գիտ. դոկտոր Ս. Վոստոկով
	ֆիզ-մաթ. գիտ. դոկտոր Վ. Միքայելյան
Առաջատար կազմակերպություն՝	Ռուսաստանի գիտությունների ակադեմիայի Վ. Ա. Ստեկլովի անվ. մաթեմատիկայի ինստիտուտի Սանկտ Պետերբուրգի մասնաճյուղ

Պաշտպանությունը կկայանա 2018թ. նոյեմբերի 6–ին ժ. 15^{00}–ին Երևանի պետական համալսարանում գործող ԲՈՀ-ի 050 մասնագիտական խորհրդի նիստում (0025, Երևան, Ալեք Մանուկյան 1):

Ատենախոսությանը կարելի է ծանոթանալ ԵՊՀ-ի գրադարանում:
Սեղմագիրն առաքվել է 2018թ. հոկտեմբերի 5–ին:

Մասնագիտական խորհրդի գիտական քարտուղար,

ֆիզ-մաթ. գիտ. դոկտոր Տ. Ն. Հարությունյան

Dissertation topic was approved at a meeting of academic council of the faculty of Mathematics and Mechanics of the Yerevan State University.

Supervisor:	Doctor of phys-math. sciences V. Atabekyan
Official opponents:	Doctor of phys-math. sciences S. Vostokov
	Doctor of phys-math. sciences V. Mikaelian
Leading organization:	St. Petersburg Department of Steklov Mathematical institute of Russian Academy of Sciences

Defense of the thesis will be held at the meeting of the specialized council 050 of HAC of Armenia at Yerevan State University on November 6, 2018 at 15^{00} (0025, Yerevan, A.Manoogian str. 1).

The thesis can be found in the library of the YSU.
Synopsis was sent on October 5, 2018.

Scientific secretary of specialized council,

Doctor of mathematics T. N. Harutyunyan

General characteristics of the work

Relevance of the theme. Let L/K be a finite Galois extension of fields with Galois group G. Then L can be regarded as a module over the group ring $K[G]$, where K acts by multiplication and G acts in a natural way. The normal basis theorem implies that this module is in fact cyclic, namely there is an isomorphism $L \cong K[G]$ of $K[G]$-modules. Similarly one may consider the $\mathbb{Z}[G]$-module L^*, but this time its structure is rather complicated and is not well studied even if the fields L and K are "good" enough.
If K and L are finite extensions of $\mathbb{Q}_p$, then the ring of integers $\mathcal{O}_L$ as well as all the fractional ideals $\mathfrak{p}_L^n, n \in \mathbb{Z}$ of the field L are $\mathcal{O}_K[G]$-modules and one might ask a question concerning their structure . In connection with this *S.V.Vostokov* proved in [1] that if L/K is an abelian p-extension with Galois group G and K does not contain a primitive p-th root of unity, then in the field L there exist ideals which are decomposable as $\mathcal{O}_K[G]$-modules if and only if the ramification index $e(L/K)$ divides the different $\mathfrak{D}_{L/K}$.
The multiplicative case was studied in a series of articles [2–6]. The starting point of our investigations was the article [4] by *D.K.Faddeed*, where he considered the case of cyclic p-extension L/K with Galois group $G = \langle\sigma\rangle$, such that K contains a primitive p-th root of unity ζ_p. It was proved that there is an isomorphism of $R = \mathbb{F}_p[G]$-modules

$$L^*/L^{*p} \cong \begin{cases} R^n \oplus R/(\sigma-1)^2, & \zeta_p \in N_{L/K}(L^*) \\ R^n \oplus R/(\sigma-1) \oplus R/(\sigma-1), & \text{otherwise} \end{cases}$$

Moreover, the author provided a canonical method for selecting the generating elements in a way that they satisfy certain requirements concerning *Hasse's* norm residue symbol.
In a similar way one may consider E/E^p as an R-module, where $E = 1 + \mathfrak{p}_L$ is the group of principal units in L. In the article [7] we considered this module in the case of a cyclic extension L/K of degree p. It turned out that its structure depends on several factors, including the ramification of the extension L/K. The proof of this result is provided in the first paragraph of the second chapter of the thesis.
Further, *Z.I.Borevich* studied E as a $\mathbb{Z}_p[G]$-module in the case of cyclic p-extensions L/K. In this respect the following theorem concerning unramified extensions was proved in [5].

Theorem (Borevich, 1965) *If the extension L/K is unramified and the fields L and K have the same irregularity degree $s \geq 1$, then for the $\mathbb{Z}_p[G]$-module E there exist a system of generating elements $\theta_1, ..., \theta_{n-1}, \xi, \omega$ with the unique defining relation $\xi^{p^s} = \omega^{\sigma-1}$, where $n = [K : \mathbb{Q}_p]$.*

With the development of the theory of formal groups it became clear that the considered additive and multiplicative cases are special cases of a more general construction. Namely, let $M/L, L/K, K/\mathbb{Q}_p$ be finite extensions with M/L Galois. If F is a one dimensional formal group law over the ring $\mathcal{O}_K$ then one can introduce a new operation on the maximal ideal $\mathfrak{p}_M$ according to the rule $x \underset{F}{+} y = F(x, y)$. From the axioms of a formal group it follows that the structure obtained is in fact an abelian group. It is denoted by $F(\mathfrak{p}_M)$. Now the task is to study $F(\mathfrak{p}_M)$ as a $\mathrm{Gal}(M/L)$-module, more precisely as a $\mathrm{End}_{\mathcal{O}_K}(F)[\mathrm{Gal}(M/L)]$-module, where $\mathrm{End}_{\mathcal{O}_K}(F)$ is the ring of endomorphisms of the formal group F. If $F = \mathbb{G}_a, F(x, y) = x + y$ is the additive formal group, then $F(\mathfrak{p}_M)$

is simply the ideal $\mathfrak{p}_M$ and $\mathrm{End}_{\mathcal{O}_K}(F) = \mathcal{O}_K$. This is precisely the additive case considered above. Similarly, if $K = \mathbb{Q}_p$ and $F = \mathbb{G}_m$ is the multiplicative formal group given by the law $F(x, y) = x + y + xy$, then $\mathrm{End}_{\mathcal{O}_K}(F) = \mathbb{Z}_p$ and $F(\mathfrak{p}_M) \cong E, x \mapsto 1 + x$ as $\mathbb{Z}_p[\mathrm{Gal}(M/L]$-modules, where $E = 1 + \mathfrak{p}_M$ is the group of principal units in M. This one is the multiplicative case considered earlier. Moreover, if we want a particularly large endomorphism group, we arrive at the so-called *Lubin-Tate* formal groups F. A slight modification of *Borevich's* theorem was proved in the paper [8], where the authors studied the structure of $F(\mathfrak{p}_M)$ in the case of a *Lubin-Tate* formal group F. In the paper [9] *T.Honda* introduced a new method of constructing formal groups which were later named in his honor. They appeared to be generalizations of *Lubin-Tate* formal groups due to the classification theorems of *O.V.Demchenko*, proved in [10]. Thanks to the tight connection between these two types of formal groups it became possible to generalize the results already proved for *Lubin-Tate* formal groups to the case of *Honda* formal groups. In particular we generalized the latter result to the case of *Honda* formal groups in the second paragraph of the second chapter of the thesis (See also [13]).
Third chapter of the thesis contains some of our results on arithmetic sequences, namely the sequences whose terms are integers. We would like to mention a result related to the *Fermat* sequence defined by the formula $a_n = 2^{2^n} + 1$ for all n. It is known that any two distinct terms of this sequence are coprime. We were interested in whether the result would remain true if we replaced 1 with an odd number d. The research answered this question in the negative (see Theorem 3.3). The second section contains the proof of a theorem concerning asymptotic estimates of the divisor function.

The aim of the thesis:

1. To study the structure of the $\mathbb{F}_p[\mathrm{Gal}(K/k)]$-module $V = E/E^p$ in the case of cyclic extensions K/k of local fields of prime degree p, where E is the group of principal units in K.

2. To find generating elements and defining relations for the $\mathcal{O}_{K_0}[\mathrm{Gal}(M/L)]$-module $F(\mathfrak{p}_M)$ in the case of unramified cyclic p-extensions M/L of local fields and Honda formal groups $F/\mathcal{O}_K$ relative to the unramified extension K/K_0.

3. To prove that a sequence $A = (a_n)_{n=1}^{\infty}$ of positive integers is distinguished as soon as it satisfies one of the conditions listed below

   - $\liminf\limits_{n\to\infty} \dfrac{\ln(\ln(a_n))}{\ln(n)} = 0$ and A is increasing.
   - $\lim\limits_{n\to\infty} a_n = \infty$ and there is a sequence $(b_n)_{n=1}^{\infty}$ such that $\gcd(a_k, a_{k+l}) < b_l$ for all k and l.

4. To prove that if d is any integer different from 1, then for any $M > 0$ there exist distinct positive integers m and n such that $\gcd(2^{2^m} + d, 2^{2^n} + d) > M$.

5. To prove that if $A = (n_k)_{k=1}^{\infty}$ is a sequence of positive integers each term of which is a power of a prime, namely $n_k = p_k^{m_k}$, where p_k is prime for all k, then

- If $\mu > 0$, then $T_{n_k}(\mu) \to \infty$, as $m_k \to \infty$.
- If $1 \le \mu < \theta^{-1}$, then $T_{n_k}(\mu) \to \infty$, as $n_k \to \infty$.

where

$$T_n(\mu) = (\tau(n))^{-1} \max_{1 \le t \le [n^{1/\mu}]} \{\tau(n+t)\}, \; n = 1, 2, ...,$$

$$\theta = \inf\{\lambda > 0 | D(x) = x \ln x + (2\gamma - 1)x + O(x^{\lambda})\}$$

and $D(x) = \sum_{n \le x} \tau(n)$ is the summatory function of τ.

The methods of investigations. In the thesis we apply methods and use the results obtained on the basis of the theory of local fields, formal groups and Galois modules, as well as other known methods developed by the St. Petersburg School of Algebraic Number Theory.

Scientific innovation. All results are new, with the exception for Theorem 3.1, which was previously proved in [12] for almost injective sequences. We would like to stress that our result is independent of [12].

Practical and theoretical value. The results of the work have theoretical character. The results of the thesis can be used in the study of formal modules defined in extensions of local fields, as well as in the investigations regarding arithmetic sequences.

Approbation of the results. The obtained results were presented at the Research Seminar of Constructive Class Field Theory of St. Petersburg State University, 2016-2018 as well were scheduled for presentation at the International Conference On Number Theory which took place in Palagna, Lithuania from 09 to 15 September, 2018.

Publications. The main results of the thesis have been published in 3 scientific articles and one preprint. The list of the articles is given at the end of the Synopsis.

The structure and the volume of the thesis. The thesis consists of introduction, 3 chapters and a list of references. The number of references is 35. The volume of the thesis is 76 pages.

The main results of the thesis

Chapter 1. The aim of the first chapter is to introduce all the necessary concepts concerning local fields, formal groups and G-modules as well as to develop basic tools necessary for further understanding of the text. Whenever the proof of a statement is omitted, the exact reference to the relevant place in the literature is given.

Chapter 2. §1. The reduced group of principal units as Galois module Let p be a rational prime number, k be a finite extension of $\mathbb{Q}_p$ of degree n containing a primitive p-th root of unity ζ_p and let K/k be a cyclic extension of degree $L = p^m$ for some positive integer m. Let us fix a generator σ of the Galois group $G = \mathrm{Gal}(K/k)$. We denote by Γ and γ the norm group of the extension K/k and k_1/k respectively, where k_1 is the unique subfield of K of degree p over k. Observe that the group K^*/K^{*p} can be regarded as a multiplicatively written $\mathbb{F}_p$-vector space. In the article [4] the following theorem is proved

Theorem. (Faddeev, 1959) *In the group K^* there is an almost normal basis modulo K^{*p} of the form*

$$\{A_1, A_1^{\sigma}, ..., A_1^{\sigma^{L-1}}, A_2, A_2^{\sigma}, ..., A_2^{\sigma^{L-1}}, ..., A_n, A_n^{\sigma}, ..., A_n^{\sigma^{L-1}}, A_0, b\}$$

with the following cases
Case 1 *$(\zeta_p \notin \Gamma)$: $b \in k^* \setminus \gamma$ is any element and $A_0 \in K^*$ is any element satisfying $\frac{\sigma(A_0)}{A_0} \in bK^{*p}$.*
Case 2 *$(\zeta_p \in \Gamma)$: $b \in k^* \setminus \gamma$ is any element and $A_0 \in K^*$ is an element, satisfying $\frac{\sigma(A_0)}{A_0} = B_0^p$, where $B_0 \in K^*$ satisfies $N_{K/k}(B_0) = \zeta_p$.*

Let $R = \mathbb{F}_p[G]$ denote the group ring over G. Note that from the results of the theorem follows the isomorphism of R-modules $K^*/K^{*p} \cong R^n \oplus R/(\sigma-1)^2$ in the first case and $K^*/K^{*p} \cong R^n \oplus R/(\sigma-1) \oplus R/(\sigma-1)$ in the second case. The idea of the proof is as follows.
The linear operator $v = \sigma - 1 \in R$ satisfies the conditions $v^L = \sigma^L - 1 = 0, v^{L-1} \neq 0$, so that it is nilpotent of degree L. Hence the whole space K^*/K^{*p} decomposes into a direct sum of cyclic v-invariant subspaces. Moreover, each of these subspaces is an R-module, isomorphic to $R/(\sigma-1)^k$, where k is the dimension of the corresponding subspace. What remains is to calculate the number of cyclic subspaces of each dimension k.

Observe that in exactly the same way v can be regarded as a linear operator on the space $V = E/E^p$, where $E = 1 + \mathfrak{p}_K$ is the group of principal units in K^*. In complete analogy with the previous case we study the structure of V as an R-module assuming, in addition, that K/k is a cyclic extension of degree $L = p$. According to the research everything proven in the first paragraph can be incorporated within the following table in the form of a theorem. Here u denotes a principal unit in k.

Theorem 2.1

	The extension K/k	R-module V
1.	$\zeta_p \notin k$, unramified	R^n
2.	$\zeta_p \notin k$, totally ramified	$R^{n-1} \oplus R/(\sigma-1)^{p-1} \oplus R/(\sigma-1)$
3.	$K = k(\sqrt[p]{u}), \zeta_p \in k \setminus \Gamma$	$R^n \oplus R/(\sigma-1)$
4.	$K = k(\sqrt[p]{\pi}), \zeta_p \in k \setminus \Gamma$	$R^{n-1} \oplus R/(\sigma-1)^{p-1} \oplus R/(\sigma-1)^2$
5.	$\zeta_p \in \Gamma$, unramified	$R^n \oplus R/(\sigma-1)$
6.	$K = k(\sqrt[p]{u}), \zeta_p \in \Gamma$, totally ramified	$R^n \oplus R/(\sigma-1)$
7.	$K = k(\sqrt[p]{\pi}), \zeta_p \in \Gamma$, totally ramified	$R^{n-1} \oplus R/(\sigma-1)^{p-1} \oplus R/(\sigma-1) \oplus R/(\sigma-1)$

It can be proved that in all cases any $\mathbb{F}_p$-basis of V modulo $\ker((\sigma-1)^{p-1})$ can serve as an R-basis of the corresponding free part. Moreover, it is not hard to show that in cases 3 and 6 the additional generator α can be chosen as any element from the set $U_1 \setminus \Gamma_0$, while in case 5 one may choose $\alpha = u_{\beta_0}$ (see Section 2.2.1).

§2. Honda formal group as Galois module Let p be a rational prime, $K/\mathbb{Q}_p, L/K, M/L$ be a tower of finite extensions of local fields, M/L be a Galois extension with Galois group G and F be a one dimensional formal group law over the ring $\mathcal{O}_K$. The

operation $x \underset{F}{+} y = F(x, y)$ sets a new structure of abelian group on the maximal ideal $\mathfrak{p}_M$ of the ring $\mathcal{O}_M$ which we will denote by $F(\mathfrak{p}_M)$. Taking into account the natural action of the group G on $F(\mathfrak{p}_M)$, one may consider it as an $\mathrm{End}_{\mathcal{O}_K}(F)[G]$-module, in which the multiplication by scalars from $\mathrm{End}_{\mathcal{O}_K}(F)$ is performed by the rule $f * x = f(x)$.
If F is a Lubin-Tate formal group law, then there is an injection $\mathcal{O}_K \hookrightarrow \mathrm{End}_{\mathcal{O}_K}(F)$, which enables us to regard $F(\mathfrak{p}_M)$ as an $\mathcal{O}_K[G]$-module. The structure of this module in case of multiplicative formal group $F = G_m$ and $K = \mathbb{Q}_p$ is studied in sufficient detail in [2, 3, 5]. The starting point of the current study is the following theorem of Borevich in [5].

Theorem (Borevich, 1965) *If the extension L/K is unramified and the fields L and K have the same irregularity degree $s \geq 1$, then for the $\mathbb{Z}_p[G]$-module E there exist a system of generating elements $\theta_1, ..., \theta_{n-1}, \xi, \omega$ with the unique defining relation $\xi^{p^s} = \omega^{\sigma-1}$, where $n = [K : \mathbb{Q}_p]$.*

It may seem that the group of principal units E_M has nothing to do with formal groups, but in fact it is easy to show that for the multiplicative formal group $F = G_m$ there is an isomorphism $F(\mathfrak{p}_M) \cong E_M, x \mapsto 1 + x$ of $\mathbb{Z}_p[G]$-modules.

The next stop in the course of investigations was the joint work of S.V.Vostokov and I.I.Nekrasov [8], where they generalized the aforementioned theorem to the case of Lubin-Tate formal groups. The key point in their work was the proof of the triviality of the cohomology groups $H^i(G(M/L), F(\mathfrak{p}_M)), i = 0, -1$ for unramified extensions M/L. More precisely, they managed to prove the following

Theorem (Vostokov-Nekrasov, 2014) *Suppose M/L is an unramified p-extension and F is a Lubin-Tate formal group for the prime element $\pi \in K$. Assume moreover that the fields M and L have the same irregularity degree, namely they contain a generator of $\ker[\pi^s]_F$ and do not contain a generator of $\ker[\pi^{s+1}]_F$ for some $s \geq 1$. Then for the $\mathcal{O}_K[G]$-module $F(\mathfrak{p}_M)$ there exists a system of generating elements $\theta_1, ..., \theta_{n-1}, \xi, \omega$ with the unique defining relation $[\pi^s]_F(\xi) = \omega^\sigma \underset{F}{-} \omega$, where $n = [L : K]$ and σ is a generating element of the Galois group $G = \mathrm{Gal}(M/L)$.*

In its turn, our work is devoted to the generalization of the last result to the case of Honda formal groups (See [13]). Namely, let $K_0/\mathbb{Q}_p$ be a finite extension such that K/K_0 is unramified, $\pi \in K_0$ be a uniformizer, F be a Honda formal group over $\mathcal{O}_K$ relative to the extension K/K_0 of special type $u \in \mathcal{O}_{K,\varphi}[[T]]$ and height h. Suppose K^{alg} is a fixed algebraic closure of the field K, $\mathfrak{p}_{K^{\mathrm{alg}}}$ is the valuation ideal, i.e. the set of all points in K^{alg} with positive valuation. Define $W_F^n = \ker[\pi^n]_F \subset F(\mathfrak{p}_{K^{\mathrm{alg}}})$ to be the π^n-torsion submodule and let $W_F = \bigcup\limits_{n=1}^{\infty} W_F^n$.
It is known that there is a ring embedding $\mathcal{O}_{K_0} \hookrightarrow \mathrm{End}_{\mathcal{O}_K}(F)$, which allows as to regard $F(\mathfrak{p}_M)$ as an $\mathcal{O}_{K_0}[G]$-module. In this section, using generators and defining relations we describe the structure of this module provided that certain conditions are satisfied. More, precisely we prove the following

Theorem 2.2 *If the extension M/L is unramified and $W_F \cap F(\mathfrak{p}_L) = W_F \cap F(\mathfrak{p}_M) = W_F^s$, for some $s \geq 1$, then $h \leq n$ and for the $\mathcal{O}_{K_0}[G]$-module $F(\mathfrak{p}_M)$ there exist a system of generating elements $\theta_j, \xi_i, \omega_i, 1 \leq j \leq n-h, 1 \leq i \leq h$ with the only defining relations*

$[\pi^s]_F(\xi_i) = \omega_i^\sigma \underset{F}{-} \omega_i, 1 \le i \le h.$

Chapter 3. §1. Distinguished sequences

It is known that for any non-constant polynomial P with integer coefficients there exist infinitely many primes, dividing at least one term of the sequence $(P(n))_{n=1}^{\infty}$ [11, Part 8, Ex. 108]. In this connection the following definition is given.

Definition 3.1 *A sequence $(n_k)_{k=1}^{\infty}$ of positive integers is called distinguished, if there are infinitely many primes dividing at least one term of the sequence.*

To formulate our results we need one more

Definition 3.2 *Suppose $S = \{p_1, p_2, ..., p_n\}$ is a finite set consisting of prime numbers. We define $\hat{S} = \{{p_1}^{k_1}{p_2}^{k_2} \cdot \ldots \cdot {p_n}^{k_n} | k_1, k_2, ..., k_n \in \mathbb{Z}_+\}$ and arrange the set $\hat{S}$ in increasing order to get the sequence $(n_k(S))_{k=1}^{\infty}$.*

The aim of this section is to prove the following theorems.

Theorem 3.1 $\lim\limits_{k\to\infty} \dfrac{\ln(\ln(n_k(S)))}{\ln(k)} = \dfrac{1}{n}.$

Theorem 3.2 *Suppose $(n_k)_{k=1}^{\infty}$ and $(m_k)_{k=1}^{\infty}$ are sequences of positive integers. Then the sequence $(n_k)_{k=1}^{\infty}$ is distinguished, provided the following conditions hold.*

1. $\lim\limits_{k\to\infty} n_k = \infty$

2. $\gcd(n_k, n_{k+l}) < m_l$ *for all positive integers k and l*

We also deal with sequences of the form $a_n = 2^{2^n} + d$, where $d \in \mathbb{Z}$. Recall that a Fermat number is a positive integer of the form $F_n = 2^{2^n} + 1$ for some nonnegative integer n. It is well known that $\gcd(F_k, F_l) = 1$, whenever $k \neq l$. In this way a natural question arises: Whether this result is true for sequences $a_n = 2^{2^n} + d$ for odd integers d? In this respect we prove a theorem, which gives a negative answer to this question.

Theorem 3.3 *If d is any integer different from 1, then for any $M > 0$ there exist distinct positive integers m and n such that $\gcd(2^{2^m} + d, 2^{2^n} + d) > M$.*

§2. Some analytic estimates for the divisor τ-function The function $\tau(n)$ is defined as the number of positive divisors of the given positive integer n. Let $D(x) = \sum_{n\le x} \tau(n)$ be the summatory function of τ. Dirichlet proved the asymptotic equality

$$D(x) = x \ln x + (2\gamma - 1)x + O(x^{1/2}),$$

where γ is the Euler's constant. Dirichlet's divisor problem consists of determining the smallest θ for which the error term is $O(n^{\theta+\epsilon})$ for any $\varepsilon > 0$. G.Voronoi has showed that $\theta \le 1/3$, while Hardy and Gauss proved that the error term is not $O(x^{1/4})$, and therefore $\theta \ge 1/4$. It is conjectured that $\theta = 1/4$.

For $\mu > 0$ consider the sequence

$$T_n(\mu) = (\tau(n))^{-1} \max_{1 \le t \le [n^{1/\mu}]} \{\tau(n+t)\},\ n = 1, 2, ...$$

Assume that $(n_k)_{k=1}^{\infty}$ is a sequence of positive integers such that $n_k = {p_k}^{j_k}$, where p_k is prime and $j_k \in \mathbb{N}$ for all positive integers k. The aim of this section is to prove the following theorem.

Theorem 3.4 a) *If* $\mu > 0$*, then* $T_{n_k}(\mu) \to \infty$*, as* $j_k \to \infty$*;* **b**) *If* $1 \le \mu < \theta^{-1}$*, then* $T_{n_k}(\mu) \to \infty$*, as* $n_k \to \infty$*.*

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ԱՄՓՈՓՈՒՄ

Տիգրան Լևոնի Հակոբյան

Ձևական մոդուլների սիմպլեկտիկ փարածություն

Ատենախոսությունում ստացվել են հետևյալ արդյունքները.

1. Ուսումնասիրվել է $V = E/E^p$ խմբի կառուցվածքը որպես $\mathbb{F}_p[\mathrm{Gal}(K/k)]$-մոդուլ լոկալ դաշտերի p պարզ կարգի ցիկլիկ K/k ընդլայնումների համար, որտեղ E-ն K դաշտի գլխավոր միավորների խումբն է:

2. $\mathcal{O}_{K_0}[\mathrm{Gal}(M/L)]$-մոդուլ $F(\mathfrak{p}_M)$-ի համար գտնվել են ծնիչներ և որոշիչ առնչություններ լոկալ դաշտերի M/L չճյուղավորված p-ընդլայնումների և K/K_0 չճյուղավորված ընդլայնման նկատմամբ տրված Հոնդայի $F/\mathcal{O}_K$ ձևական խմբերի դեպքում:

3. Ապացուցվել է, որ բնական թվերից կազմված $A = (a_n)_{n=1}^{\infty}$ հաջորդականությունը *պարբերակված է* , եթե բավարարվում է հետևյալ պայմաններից որևէ մեկը

   - $\liminf\limits_{n\to\infty} \dfrac{\ln(\ln(a_n))}{\ln(n)} = 0$ և A-ն աճող է:
   - $\lim\limits_{n\to\infty} a_n = \infty$ և գոյություն ունի $(b_n)_{n=1}^{\infty}$ հաջորդականություն, այնպես որ $\gcd(a_k, a_{k+l}) < b_l$ բոլոր k և l ինդեքսների համար:

4. Ապացուցվել է, որ եթե d-ն 1-ից տարբեր կամայական ամբողջ թիվ է, ապա ցանկացած $M > 0$ թվի համար գոյություն ունեն իրարից տարբեր m և n ինդեքսներ, այնպիսիք որ $\gcd(2^{2^m} + d, 2^{2^n} + d) > M$:

5. Ապացուցվել է, որ եթե բնական թվերից կազմված $A = (n_k)_{k=1}^{\infty}$ հաջորդականության կամայական անդամ ունի $n_k = p_k^{m_k}$ տեսքը, որտեղ p_k պարզ թիվ է, ապա ճշմարիտ են հետևյալ պնդումները

   - Եթե $\mu > 0$, ապա $T_{n_k}(\mu) \to \infty$, երբ $m_k \to \infty$:
   - Եթե $1 \le \mu < \theta^{-1}$, ապա $T_{n_k}(\mu) \to \infty$, երբ $n_k \to \infty$:

   որտեղ

$$T_n(\mu) = (\tau(n))^{-1} \max_{1\le t\le \left[n^{1/\mu}\right]} \{\tau(n+t)\},\ n = 1, 2, ...,$$

$$\theta = \inf\{\lambda > 0 | D(x) = x\ln x + (2\gamma - 1)x + O(x^\lambda)\}$$

   և $D(x) = \sum_{n\le x} \tau(n)$-ը τ-ֆունկցիայի գումարային ֆունկցիան է:

Заключение

Акопян Тигран Левонович

Симплектическое пространство формальных модулей

В диссертации получены следующие результаты:

1. Изучена структура $\mathbb{F}_p[\mathrm{Gal}(K/k)]$-модуля $V = E/E^p$ в случае циклических расширений K/k простой степени p локальных полей, где E группа главных единиц поля K.

2. Найдены порождающие элементы и определяющие соотношения для $\mathcal{O}_{K_0}[\mathrm{Gal}(M/L)]$-модуля $F(\mathfrak{p}_M)$ в случае неразветвленного p-расширения M/L локальных полей и формальных групп Хонды $F/\mathcal{O}_K$, заданных относительно неразветвленного расширения K/K_0.

3. Доказано, что последовательность натуральных чисел $A = (a_n)_{n=1}^{\infty}$ *отмечена*, коль скоро выполняется одно из нижеследующих условий

   - $\liminf\limits_{n\to\infty} \dfrac{\ln(\ln(a_n))}{\ln(n)} = 0$ и A возрастает.
   - $\lim\limits_{n\to\infty} a_n = \infty$ и существует последовательность $(b_n)_{n=1}^{\infty}$, такая что $\gcd(a_k, a_{k+l}) < b_l$ для всех k и l.

4. Доказано, что если d отличное от 1 целое число, то для любого $M > 0$ существуют различные индексы m и n, для которых $\gcd(2^{2^m} + d, 2^{2^n} + d) > M$.

5. Доказано, что если последовательность натуральных чисел $A = (n_k)_{k=1}^{\infty}$ такова, что каждый ее член имеет вид $n_k = p_k^{m_k}$, где p_k простое число, то справедливы следующие утверждения

   - Если $\mu > 0$, то $T_{n_k}(\mu) \to \infty$, при $m_k \to \infty$.
   - Если $1 \le \mu < \theta^{-1}$, то $T_{n_k}(\mu) \to \infty$, при $n_k \to \infty$.

   где

   $$T_n(\mu) = (\tau(n))^{-1} \max_{1\le t\le\left[n^{1/\mu}\right]} \{\tau(n+t)\},\ n = 1, 2, ...,$$

   $$\theta = \inf\{\lambda > 0 | D(x) = x\ln x + (2\gamma - 1)x + O(x^{\lambda})\}$$

   и $D(x) = \sum_{n\le x} \tau(n)$ сумматорная функция функции количества делителей τ.