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**Convergence Acceleration of Approximations
by the Modified Fourier System**

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THESIS

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Introduction

We consider approximations and interpolations by the modified trigonometric system

$$\mathcal{H} = \{\cos \pi n x : n \in \mathbb{Z}_+\} \cup \{\sin \pi(n - \frac{1}{2})x : n \in \mathbb{N}\}, x \in [-1, 1] \quad (0.1)$$

which was originally proposed by Krein [1] without investigation of its properties. Expansions by the modified trigonometric system were studied in a series of papers [2–13].

The set \mathcal{H} is an orthonormal basis of $L_2[-1, 1]$, as it consists of the eigenfunctions of the Sturm-Liouville operator

$$\mathcal{L} = -d^2/dx^2 \quad (0.2)$$

with Neumann boundary conditions $u'(1) = u'(-1) = 0$. Both, the orthogonality and density in $L_2[-1, 1]$ follow from the classical spectral theory ([14]).

Let $M_N(f, x)$ be the truncated modified Fourier series

$$M_N(f, x) = \frac{1}{2}f_0^c + \sum_{n=1}^N [f_n^c \cos \pi n x + f_n^s \sin \pi(n - \frac{1}{2})x], \quad (0.3)$$

where

$$f_n^c = \int_{-1}^1 f(x) \cos \pi n x dx, \quad f_n^s = \int_{-1}^1 f(x) \sin \pi(n - \frac{1}{2})x dx. \quad (0.4)$$

Let

$$R_N(f, x) = f(x) - M_N(f, x). \quad (0.5)$$

Obviously, for even functions on $[-1, 1]$, expansions by the modified Fourier system coincide with the expansions by the classical Fourier system

$$\mathcal{H}_{class} = \{\cos \pi n x : n \in \mathbb{Z}_+\} \cup \{\sin \pi n x : n \in \mathbb{N}\}, x \in [-1, 1]. \quad (0.6)$$

Moreover, the modified Fourier system can be derived from the other classical system \mathcal{H}^*

$$\mathcal{H}^* = \{\cos \pi n x : n \in \mathbb{Z}_+\}, x \in [0, 1] \quad (0.7)$$

by means of a change of variable.

The first results concerning the convergence of the expansions by the modified trigonometric system were appeared in the works [2–9].

Theorem 0.1 [6] *Assume $f \in C^2[-1, 1]$ and $f'' \in BV[-1, 1]$. If $|x| < 1$, then*

$$R_N(f, x) = O(N^{-2}), \quad N \rightarrow \infty. \quad (0.8)$$

Otherwise,

$$R_N(f, \pm 1) = O(N^{-1}), \quad N \rightarrow \infty. \quad (0.9)$$

As we see, expansions by the modified trigonometric system have better convergence properties for smooth odd functions on $[-1, 1]$ compared to the classical expansions([15]).

Let $f \in C^{2q+2}[-1, 1]$ and

$$A_{2k+1}(f) = (f^{(2k+1)}(1) - f^{(2k+1)}(-1)) (-1)^k, \quad k = 0, \dots, q-1, \quad (0.10)$$

and

$$B_{2k+1}(f) = (f^{(2k+1)}(1) + f^{(2k+1)}(-1)) (-1)^k, \quad k = 0, \dots, q-1. \quad (0.11)$$

By means of integration by parts and Riemann-Lebesgue lemma, we get the following asymptotic expansions for the modified Fourier coefficients

$$f_n^c = (-1)^n \sum_{k=0}^q \frac{A_{2k+1}(f)}{(\pi n)^{2k+2}} + o(n^{-2q-2}), \quad n \rightarrow \infty, \quad (0.12)$$

and

$$f_n^s = (-1)^{n+1} \sum_{k=0}^q \frac{B_{2k+1}(f)}{(\pi(n - \frac{1}{2}))^{2k+2}} + o(n^{-2q-2}), \quad n \rightarrow \infty. \quad (0.13)$$

We see that faster convergence of the modified expansions compared to the classical Fourier expansions could be explained by faster decay of coefficients f_n^s

$$f_n^s = O(n^{-2}), \quad n \rightarrow \infty \quad (0.14)$$

when f is enough smooth, but non-periodic on $[-1, 1]$. Estimate (0.14) can be also explained by a non-periodicity of the system functions $\sin \pi(n - \frac{1}{2})x$ on $[-1, 1]$. Also, we see that for

more rapid decay of the modified Fourier coefficients, the approximated function must obey the first q derivative conditions

$$f^{(2r+1)}(\pm 1) = 0, \quad r = 0, 1, \dots, q-1. \quad (0.15)$$

Under such additional requirements, the convergence rate in Theorem 0.1 could be made faster.

Theorem 0.2 [5, 6] *Assume $f \in C^{2q+2}(-1, 1)$, $f^{(2q+2)} \in BV[-1, 1]$, $q \geq 1$ and f obeys the first q derivative conditions (0.15). If $|x| < 1$, then*

$$R_N(f, x) = O(N^{-2q-2}), \quad N \rightarrow \infty. \quad (0.16)$$

Otherwise,

$$R_N(f, \pm 1) = O(N^{-2q-1}), \quad N \rightarrow \infty. \quad (0.17)$$

We see that the derivative conditions (0.15) are crucial for convergence properties of the expansions by the modified trigonometric system. Without those conditions, the convergence will remain slow. If a function doesn't obey those derivative conditions, then, application of a well-known polynomial subtraction method will correct the derivatives at the endpoints $x = \pm 1$. For the classical Fourier series this approach has a very long history (see [9, 16–22]). For the modified expansions these approach is explored in [4, 8, 9]. More specifically, we write f (see [9]) in the terms of its Lanczos representation

$$f = (f - g_k) + g_k, \quad (0.18)$$

where functions (polynomials) g_k are chosen such to satisfy the conditions

$$f^{(2r+1)}(\pm 1) = g_k^{(2r+1)}(\pm 1), \quad r = 0, \dots, k-1. \quad (0.19)$$

Since $f - g_k$ obeys the first k derivative conditions, the new approximation

$$M_N^k(f, x) = M_N(f - g_k, x) + g_k$$

will converge with the same rate as if f obeyed those conditions. This is the polynomial subtraction technique known also as Krylov-Lanczos approach ([23]). If the jumps of f are

unknown, their values can be approximated by solution of the corresponding system of linear equations (see [21]). Now, assume that even polynomials $P_k(x)$ and odd polynomials $Q_k(x)$, $k = 0, \dots, q-1$ satisfy the following conditions (see [24])

$$\begin{aligned} A_{2k+1}(P_j(x)) &= \delta_{k,j}, \quad 0 \leq k, j \leq q-1, \\ B_{2k+1}(Q_j(x)) &= \delta_{k,j}, \quad 0 \leq k, j \leq q-1. \end{aligned} \tag{0.20}$$

The first few polynomials are

$$\begin{aligned} P_0(x) &= \frac{1}{4}x^2, & P_1(x) &= \frac{1}{48}x^2(x^2 - 2), \\ Q_0(x) &= \frac{1}{2}x, & Q_1(x) &= \frac{1}{12}x(x^2 - 3). \end{aligned} \tag{0.21}$$

Now, let F be defined as follows

$$F(x) = f(x) - \sum_{k=0}^{q-1} A_{2k+1}(f)P_k(x) - \sum_{k=0}^{q-1} B_{2k+1}(f)Q_k(x). \tag{0.22}$$

Then, F obeys the first q derivative conditions (0.15). Now, if we denote

$$M_N^q(f, x) = M_N(F, x) + \sum_{k=0}^{q-1} A_{2k+1}(f)P_k(x) + \sum_{k=0}^{q-1} B_{2k+1}(f)Q_k(x), \tag{0.23}$$

Theorem 0.2 will be valid for approximation $M_N^q(f, x)$ without derivative conditions if the exact values of A_{2k+1} and B_{2k+1} , $k = 0, \dots, q-1$ are known. Otherwise, they can be approximated by a solution of system of linear equations (see [24]). Let

$$R_N^q(f, x) = f(x) - M_N^q(f, x). \tag{0.24}$$

We present the next three theorems for further comparisons. They explore, the convergence of the expansions by the modified trigonometric system in different frameworks. Compared to Theorems 0.1 and 0.2, the exact constants of the asymptotic errors are shown explicitly. The first theorem considers L_2 -convergence. Next two theorems describe the pointwise convergence of the modified Fourier expansions on $x \in (-1, 1)$ and at the endpoints $x = \pm 1$, respectively.

Theorem 0.3 [10] *Let $f \in C^{2q+1}[-1, 1]$, $q \geq 0$ and $f^{(2q+1)} \in BV[-1, 1]$. Then, the following estimate holds*

$$\lim_{N \rightarrow \infty} N^{2q+\frac{3}{2}} \|R_N^q(f, x)\|_{L_2} = \frac{1}{\pi^{2q+2} \sqrt{4q+3}} \sqrt{A_{2q+1}^2(f) + B_{2q+1}^2(f)}. \tag{0.25}$$

Theorem 0.4 [9] Let $f \in C^{2q+2}(-1, 1)$, $q \geq 0$ and $f^{(2q+2)} \in BV[-1, 1]$. Then, the following estimate holds $x \in (-1, 1)$ as $N \rightarrow \infty$

$$\begin{aligned} R_N^q(f, x) &= \frac{(-1)^{N+1}}{2\pi^{2q+2}N^{2q+2} \cos \frac{\pi x}{2}} \\ &\times (A_{2q+1}(f) \cos \pi(N + 1/2)x - B_{2q+1}(f) \sin \pi Nx) \\ &+ o(N^{-2q-2}). \end{aligned} \quad (0.26)$$

Theorem 0.5 [6] Let $f \in C^{2q+2}[-1, 1]$, $q \geq 0$ and $f^{(2q+2)} \in BV[-1, 1]$. Then,

$$\begin{aligned} R_N^q(f, \pm 1) &= \frac{1}{\pi^{2q+2}(2q+1)N^{2q+1}} (A_{2q+1}(f) \pm B_{2q+1}(f)) \\ &+ o(N^{-2q-1}). \end{aligned} \quad (0.27)$$

This thesis consists of five sections. Sections 1-3 explore rational approximations by the modified trigonometric system. They reproduce the results of papers [10–12]. Sections 4 and 5 study interpolations by the modified trigonometric system. They reproduce the results of paper [13].

Sections 1-3 consider rational approximations by the modified trigonometric system. Consider a finite sequence of real numbers $\theta = \{\theta_k\}_{k=1}^p$, $p \geq 1$ and by $\Delta_n^k(\theta, \hat{f})$, $\hat{f} = \{f_n\}_{n=1}^\infty$ denote the following generalized finite differences

$$\begin{aligned} \Delta_n^0(\theta, \hat{f}) &= f_n, \\ \Delta_n^k(\theta, \hat{f}) &= \Delta_n^{k-1}(\theta, \hat{f}) + \theta_k \Delta_{n-1}^{k-1}(\theta, \hat{f}), \quad k \geq 1. \end{aligned} \quad (0.28)$$

By $\Delta_n^k(\hat{f})$, we denote the classical finite differences which correspond to generalized differences $\Delta_n^k(\theta, \hat{f})$ with $\theta \equiv 1$. It is easy to verify that

$$\Delta_n^k(\hat{f}) = \sum_{\ell=0}^k \binom{k}{\ell} f_{n-\ell}. \quad (0.29)$$

Let

$$R_N(f, x) = R_N^{\cos}(f, x) + R_N^{\sin}(f, x), \quad (0.30)$$

where

$$R_N^{\cos}(f, x) = \sum_{n=N+1}^{\infty} f_n^c \cos \pi n x, \quad (0.31)$$

and

$$R_N^{\sin}(f, x) = \sum_{n=N+1}^{\infty} f_n^s \sin \pi(n - \frac{1}{2})x. \quad (0.32)$$

Consider two sequences of real numbers $\theta^c = \{\theta_k^c\}_{k=1}^p$ and $\theta^s = \{\theta_k^s\}_{k=1}^p$. Let $\hat{f}^s = \{f_n^s\}_{n=1}^\infty$ and $\hat{f}^c = \{f_n^c\}_{n=0}^\infty$. Let $\mu_j(k, \theta)$ be defined by the following identities

$$\prod_{j=1}^k (1 + \theta_j x) = \sum_{j=0}^k \mu_j(k, \theta) x^j, \quad k = 1, \dots, p. \quad (0.33)$$

By means of sequential Abel transformations (see details in [10]), we derive the following expansions of the errors (0.31) and (0.32)

$$\begin{aligned} R_N^{\cos}(f, x) &= - \sum_{k=1}^p \frac{\theta_k^c \Delta_N^{k-1}(\theta^c, \hat{f}^c)}{\prod_{r=1}^k (1 + 2\theta_r^c \cos \pi x + (\theta_r^c)^2)} \\ &\quad \times \sum_{j=0}^k \mu_j(k, \theta^c) \cos \pi(N + 1 - j)x + R_{N,p}^{\cos}(f, \theta^c, x), \end{aligned} \quad (0.34)$$

and

$$\begin{aligned} R_N^{\sin}(f, x) &= - \sum_{k=1}^p \frac{\theta_k^s \Delta_N^{k-1}(\theta^s, \hat{f}^s)}{\prod_{r=1}^k (1 + 2\theta_r^s \cos \pi x + (\theta_r^s)^2)} \\ &\quad \times \sum_{j=0}^k \mu_j(k, \theta^s) \sin \pi(N + \frac{1}{2} - j)x + R_{N,p}^{\sin}(f, \theta^s, x), \end{aligned} \quad (0.35)$$

where

$$\begin{aligned} R_{N,p}^{\cos}(f, \theta, x) &= \frac{1}{2 \prod_{k=1}^p (1 + \theta_k e^{i\pi x})} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta, \hat{f}^c) e^{i\pi n x} \\ &\quad + \frac{1}{2 \prod_{k=1}^p (1 + \theta_k e^{-i\pi x})} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta, \hat{f}^c) e^{-i\pi n x}, \end{aligned} \quad (0.36)$$

and

$$\begin{aligned} R_{N,p}^{\sin}(f, \theta, x) &= \frac{e^{-\frac{i\pi x}{2}}}{2i \prod_{k=1}^p (1 + \theta_k e^{i\pi x})} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta, \hat{f}^s) e^{i\pi n x} \\ &\quad - \frac{e^{\frac{i\pi x}{2}}}{2i \prod_{k=1}^p (1 + \theta_k e^{-i\pi x})} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta, \hat{f}^s) e^{-i\pi n x}. \end{aligned} \quad (0.37)$$

These expansions lead to the following modified-trigonometric-rational (MTR-) approximations

$$\begin{aligned} M_{N,p}(f, \theta^c, \theta^s, x) &= M_N(f, x) - \sum_{k=1}^p \frac{\theta_k^c \Delta_N^{k-1}(\theta^c, \hat{f}^c)}{\prod_{r=1}^k (1 + 2\theta_r^c \cos \pi x + (\theta_r^c)^2)} \\ &\quad \times \sum_{j=0}^k \mu_j(k, \theta^c) \cos \pi(N + 1 - j)x - \\ &\quad - \sum_{k=1}^p \frac{\theta_k^s \Delta_N^{k-1}(\theta^s, \hat{f}^s)}{\prod_{r=1}^k (1 + 2\theta_r^s \cos \pi x + (\theta_r^s)^2)} \\ &\quad \times \sum_{j=0}^k \mu_j(k, \theta^s) \sin \pi(N + \frac{1}{2} - j)x, \end{aligned} \quad (0.38)$$

with the error

$$\begin{aligned} R_{N,p}(f, \theta^c, \theta^s, x) &= f(x) - M_{N,p}(f, \theta^c, \theta^s, x) \\ &= R_{N,p}^{cos}(f, \theta^c, x) + R_{N,p}^{sin}(f, \theta^s, x). \end{aligned} \quad (0.39)$$

Similar to (0.23), we can apply the polynomial correction approach to the rational approximations

$$M_{N,p}^q(f, \theta^c, \theta^s, x) = M_{N,p}(F, \theta^c, \theta^s, x) + \sum_{k=0}^{q-1} A_{2k+1}(f)P_k(x) + \sum_{k=0}^{q-1} B_{2k+1}(f)Q_k(x), \quad (0.40)$$

with error

$$R_{N,p}^q(f, \theta^c, \theta^s, x) = f(x) - M_{N,p}^q(f, \theta^c, \theta^s, x). \quad (0.41)$$

A crucial step for realization of the rational approximations is determination of parameters θ^c and θ^s . Different approaches are known for solution of this problem (see [25–32]). In general, appropriate determination of these parameters should lead to rational approximations with improved accuracy compared to the classical ones in case of smooth f . However, the rational approximations are essentially non-linear in the sense that

$$M_{N,p}(f + g, \theta^c, \theta^s, x) \neq M_{N,p}(f, \theta^c, \theta^s, x) + M_{N,p}(g, \theta^c, \theta^s, x) \quad (0.42)$$

as for each approximation we need to determine its own θ^c and θ^s vectors.

In [10, 11], those parameters were determined from the following systems of equations

$$\Delta_n^p(\theta^c, \hat{f}^c) = 0, \quad n = N, N - 1, \dots, N - p + 1, \quad (0.43)$$

and

$$\Delta_n^p(\theta^s, \hat{f}^s) = 0, \quad n = N, N - 1, \dots, N - p + 1, \quad (0.44)$$

which led to the Fourier-Pade type approximations ([25]) with better convergence for smooth functions compared to the expansions by the modified trigonometric system ([10, 11]). We call those approximations as modified Fourier-Pade (MFP-) approximations. It is a complex approach as parameters θ^c , θ^s depend on N and systems (0.43) and (0.44) must be solved for each N .

Papers ([11, 12]) consider simpler alternative approach for smooth functions, assuming that θ^s and θ^c are determined as follows

$$\theta_k^c = 1 - \frac{\tau_k^c}{N}, \quad \theta_k^s = 1 - \frac{\tau_k^s}{N}, \quad \tau_k^c \neq 0, \tau_k^s \neq 0, \quad k = 1, \dots, p, \quad (0.45)$$

with $\tau^c = \{\tau_1^c, \dots, \tau_p^c\}$ and $\tau^s = \{\tau_1^s, \dots, \tau_p^s\}$ independent of N . Actually, we take into consideration only the first two terms of the asymptotic expansions of $\theta_k = \theta_k(N)$ in terms of $1/N$. Although, parameters θ^c and θ^s in (0.45) depend on N , we need only to determine τ^c and τ^s which are independent of N . Hence, this approach is less complex than the modified Fourier-Pade approximations.

Section 1 considers convergence of the modified Fourier-Pade approximations in different frameworks. Theorem 1.1 explores the pointwise convergence for $|x| < 1$. It shows the exact constant of the asymptotic error of the MFP-approximations when $|x| < 1$ is fixed. The convergence rate of $R_{N,p}^q$ is $O(N^{-2q-2p-2})$ as $N \rightarrow \infty$. Compared to Theorem 0.4, the improvement in convergence rate is by factor $O(N^{2p})$. However, for the modified expansions, we require less smoothness than for the MFP-approximations. Theorem 1.2 proves similar result at $x = \pm 1$. The convergence rate of $R_{N,p}^q$ is $O(N^{-2q-1})$ as $N \rightarrow \infty$. Comparison with Theorem 0.5 shows that the expansions by the modified Fourier system and the MFP-approximations have the same convergence rates at the endpoints $x = \pm 1$. However, comparison of the corresponding constants $h_{p,q}$ and $h_{0,q} = 1$ shows that the MFP-approximations are much more accurate than the classical expansions (see Table 1.1) also for $x = \pm 1$.

This section also deals with the L_2 -convergence of the MFP-approximations. Theorem 1.3 shows the exact constant of the asymptotic L_2 -error. Comparison of Theorems 0.3 and 1.3 shows that the classical expansions and the MFP-approximations have the same convergence rates $O(N^{-2q-3/2})$ in the L_2 -norm. However, comparison of the corresponding constants $c_{p,q}$ and $c_{0,q} = 1$ shows that the MFP-approximations are asymptotically more accurate (see Table 1.3).

Sections 2 and 3 consider the convergence of the MTR-approximations with parameters θ^c and θ^s defined by (0.45). In general, we derive the exact estimates for the main terms of asymptotic errors without specifying parameters τ^c and τ^s . Then, we determine the optimal

values of parameters which vanish or minimize the main terms of asymptotic errors and lead to approximations with substantially better pointwise convergence rates. We found that optimal values of parameters τ_k^c and τ_k^s , $k = 1, \dots, p$ are the roots of some polynomials depending on p and q , where q indicates the number of zero derivatives in (0.15). Moreover, the choice of optimal parameters depends on the parity of p and also on the location of x , whether $|x| < 1$ or $x = \pm 1$.

Section 2 considers the convergence of the optimal MTR-approximations on $|x| < 1$. Theorem 2.1 explores the pointwise convergence without specifying the choice of τ^c and τ^s . It shows that the convergence rate is $O(N^{-2q-p-2})$ as $N \rightarrow \infty$. Compared to Theorem 0.4, the improvement in convergence rate is by factor $O(N^p)$. Theorem 2.3 gives the optimal choice for parameters τ^c and τ^s when $|x| < 1$ and p is odd. If $\tau_k^c = \tau_k^s$, $k = 1, \dots, p$ are the roots of the generalized Laguerre polynomial $L_p^{(2q+1)}(x)$ then, the rational approximations have convergence rate $O(N^{-2q-p-\lceil \frac{p+1}{2} \rceil - 2})$ with improvement by factor $O(N^{\lceil \frac{p+1}{2} \rceil})$ compared to non-optimal choice of parameters (Theorem 2.1). The improvement is by factor $O(N^{\lceil \frac{p+1}{2} \rceil + p})$ compared to the expansions by the modified Fourier system (Theorem 0.4). In case of even p (see Theorem 2.4), possible selection set of optimal parameters is wider. If for a given p and q the following polynomial

$$\sum_{k=0}^p \binom{p}{k} \frac{1 + c_1(p-k)}{(2q+1+k)!} (-1)^k x^k \quad (0.46)$$

has only nonzero and real-valued roots $x = z_k$, $k = 1, \dots, p$ then, selection $\tau_k^s = \tau_k^c = z_k$ provides with better convergence rate $O(N^{-2q-p-\lfloor \frac{p}{2} \rfloor - 2})$ with improvement by factor $O(N^{\lfloor \frac{p}{2} \rfloor})$. Improvement is by factor $O(N^{\lfloor \frac{p}{2} \rfloor + p})$ compared to the expansions by the modified Fourier system (Theorem 0.4). When $c_1 = 0$, the roots of (0.46) coincide with the roots of the Laguerre polynomial $L_p^{(2q+1)}(x)$. When $c_1 = -1/(2q+p+1)$, the roots coincide with the ones of $L_p^{(2q)}(x)$. In both cases all roots are positive.

Theorem 2.5 of Section 2 explores the L_2 -error of the MTR-approximations without specifying the choice of the corresponding parameters. First, it derives the exact constant of the asymptotic L_2 -error. Then, parameters are selected such to minimize (numerically) the mentioned asymptotic constant. We call these approximations as L_2 -minimal MTR-approximations.

Table 2.1 shows that the latests have better asymptotic L_2 -accuracy compared to the MFP-approximations.

Section 3 explores the pointwise convergence at $x = \pm 1$. Theorem 3.1 imparts the convergence rate $O(N^{-2q-1})$ without specifying the choice of parameters. Comparison with Theorem 0.5 shows no improvement. Moreover, as our experiments show (see Figure 3.1), rational approximations without reasonable selection of parameters can perform worse at $x = \pm 1$ compared to the expansions by the modified Fourier system. Theorem 3.3 finds the optimal values of parameters for odd p . It proves that the best accuracy could be achieved when parameters $\tau_k^s = \tau_k^c$ are the roots of the generalized Laguerre polynomial $L_p^{(2q)}(x)$. For that choice, the convergence rate is $O(N^{-2q-\lceil \frac{p+1}{2} \rceil - 1})$ with improvement by factor $O(N^{\lceil \frac{p+1}{2} \rceil})$ compared to the modified Fourier expansions (Theorem 0.5) and the MFP-approximations (Theorem 1.2). In case of even p , Theorem 3.4 outlines the set of optimal parameters. If for a given p and q the following polynomial

$$\sum_{k=0}^p \binom{p}{k} \frac{1 + d_1(p-k)}{(2q+k)!} (-1)^k x^k \quad (0.47)$$

has only real-valued and non-zero roots $x = z_k$ for some d_1 then, selection $\tau_k^c = \tau_k^s = z_k$ will provide with convergence rate $O(N^{-2q-\lceil \frac{p}{2} \rceil - 1})$ with improvement by factor $O(N^{\lceil \frac{p}{2} \rceil})$ compared to the modified Fourier expansions and MFP-approximations. When $d_1 = 0$ or $d_1 = -1/(2q+p)$, the roots coincide with the ones of $L_p^{(2q)}(x)$ and $L_p^{(2q-1)}(x)$, respectively.

Sections 4 and 5 deal with interpolations by the modified trigonometric system. They explore the convergence of the modified interpolations in different frameworks: pointwise and L_2 -convergence. In each case, we derive exact constants of the asymptotic errors and provide comparisons with the classical trigonometric interpolation which shows better convergence properties of the modified interpolation for odd functions.

The modified interpolation was introduced in [13]. It is easy to verify that the modified trigonometric system can be rewritten more compactly

$$\mathcal{H} = \{\varphi_n(x) : n \in \mathbb{Z}_+\}, \quad (0.48)$$

where

$$\varphi_0(x) = \frac{1}{\sqrt{2}}, \quad \varphi_n(x) = \frac{1}{2} \left((-1)^n e^{\frac{i\pi n x}{2}} + e^{-\frac{i\pi n x}{2}} \right), \quad n \in \mathbb{N}. \quad (0.49)$$

Then, we write the interpolation as follows

$$\mathcal{I}_N(f, x) = \sum_{n=0}^{2N} \check{f}_n^m \varphi_n(x), \quad (0.50)$$

where

$$\check{f}_n^m = \frac{2}{2N+1} \sum_{k=-N}^N f(x_k) \overline{\varphi_n(x_k)}, \quad x_k = \frac{2k}{2N+1}, \quad k = 0, \pm 1, \dots, \pm N. \quad (0.51)$$

Let

$$r_N(f, x) = f(x) - \mathcal{I}_N(f, x). \quad (0.52)$$

Both, the condition of interpolation and exactness on \mathcal{H} follow from the discrete orthogonality of the modified trigonometric system for the grid x_k

$$\frac{2}{2N+1} \sum_{n=0}^{2N} \varphi_n(x_k) \overline{\varphi_n(x_s)} = \delta_{k,s}, \quad |k|, |s| \leq N, \quad (0.53)$$

and

$$\frac{2}{2N+1} \sum_{k=-N}^N \varphi_n(x_k) \overline{\varphi_m(x_k)} = \delta_{n,m}, \quad 0 \leq m, n \leq 2N. \quad (0.54)$$

When f is a real-valued function, then the modified interpolation could be rewritten as follows

$$\mathcal{I}_N(f, x) = \frac{1}{2} \check{f}_0^c + \sum_{n=1}^N \check{f}_n^c \cos \pi n x + \sum_{n=1}^N \check{f}_n^s \sin \pi (n - \frac{1}{2}) x, \quad (0.55)$$

where

$$\check{f}_0^c = \frac{2}{2N+1} \sum_{k=-N}^N f(x_k), \quad \check{f}_n^c = \frac{2}{2N+1} \sum_{k=-N}^N f(x_k) \cos \pi n x_k, \quad (0.56)$$

and

$$\check{f}_n^s = \frac{2}{2N+1} \sum_{k=-N}^N f(x_k) \sin \pi (n - \frac{1}{2}) x_k. \quad (0.57)$$

This form should be more convenient for analysis when f is either odd or even on $[-1, 1]$.

Similar to (0.23), we can apply the polynomial correction approach to the modified interpolation

$$\mathcal{I}_N^q(f, x) = \mathcal{I}_N(f, x) + \sum_{k=0}^{q-1} A_{2k+1}(f) P_k(x) + \sum_{k=0}^{q-1} B_{2k+1}(f) Q_k(x), \quad (0.58)$$

with error

$$r_N^q(f, x) = f(x) - \mathcal{I}_N^q(f, x). \quad (0.59)$$

Section 4 studies the L_2 -convergence of the modified interpolation. Theorem 4.1 shows that the L_2 -convergence rate of r_N^q is $O(N^{-2q-3/2})$ as $N \rightarrow \infty$. We see that the modified interpolation has the same convergence rate as expansions by the modified trigonometric system. When $q = 0$, Theorem 4.1 shows convergence rate $O(N^{-\frac{3}{2}})$ in the L_2 -norm. The classical interpolation

$$\mathcal{I}_N^{classic}(f, x) = \sum_{n=-N}^N \check{f}_n e^{i\pi n x}, \quad (0.60)$$

where

$$\check{f}_n = \frac{1}{2N+1} \sum_{k=-N}^N f(x_k) e^{-i\pi n x_k} \quad (0.61)$$

has convergence rate $O(N^{-\frac{1}{2}})$ in the L_2 -norm for odd functions on $[-1, 1]$ (see [33]). Hence, the improvement is by factor $O(N)$.

Section 5 explores the pointwise convergence of the modified interpolation. Theorem 5.1 shows the exact constant of the asymptotic error when $|x| < 1$ is fixed. The convergence rate of r_N^q is $O(N^{-2q-3})$ which is better than the convergence rate of the expansions by the modified trigonometric system and improvement is by factor $O(N)$. When $q = 0$, Theorem 5.1 implies the convergence rate $O(N^{-3})$ as $N \rightarrow \infty$. The classical interpolation (see [33]) has convergence rate $O(N^{-1})$ for the grid same grid x_k . Hence, improvement is by factor $O(N^2)$. Theorem 5.2 reveals the exact constant of the asymptotic error when $x = \pm 1$. It shows that the convergence rate of r_N^q is $O(N^{-2q-1})$ which is the same as for the convergence rate of the expansions by the modified trigonometric system. When $q = 0$, Theorem 5.2 shows convergence rate $O(1/N)$. In this case, as $f(1) \neq f(-1)$, the classical interpolation doesn't converge at the endpoints. Hence, the modified interpolations have better convergence rate at the endpoints with improvement by factor $O(N)$.

Modified Expansions and Interpolations

Sections 1-3 consider convergence acceleration of the modified Fourier expansions by rational corrections which lead to modified-trigonometric-rational (MTR-) approximations. The rational corrections contain some unknown parameters. We define those parameters differently. The first approach leads to the modified Fourier-Pade (MFP-) approximations (see (0.43) and (0.44)). The second approach is based on (0.45), where the values of parameters τ^c and τ^s are determined optimally to provide better convergence properties. Section 1 explores the pointwise and L_2 -norm convergence of the MFP-approximations. Section 2 considers convergence of the optimal MTR-approximations on $(-1, 1)$ and Section 3 at the endpoints $x = \pm 1$.

Sections 4, 5 introduce interpolations which are exact on the modified trigonometric system and study convergence in different frameworks: pointwise and L_2 -convergence. Section 4 explores the convergence in the L_2 -norm. Section 5 studies the pointwise convergence on $|x| < 1$ and at $x = \pm 1$. In each case, we derive the exact constants for the asymptotic errors and perform comparison with the corresponding results of the classical interpolations with the same uniform grid $x_k = \frac{2k}{2N+1}$, $|k| \leq N$ on $[-1, 1]$.

1. Convergence of the MFP-approximations

In this section, we explore convergence of the MFP-approximation in different frameworks. The first part of the section is devoted to the pointwise convergence on $|x| < 1$. In the second part, we study the convergence at $x = \pm 1$ and in the L_2 -norm. We assume that parameters θ^c and θ^s are defined by (0.43) and (0.44), respectively.

Let

$$\prod_{k=1}^p (1 + \theta_k x) = \sum_{k=0}^p \gamma_k(\theta) x^k. \quad (1.1)$$

Systems (0.43) and (0.44) can be reformulated as linear systems of equations with unknowns

$\gamma_k(\theta^c)$ and $\gamma_k(\theta^s)$ as follows

$$\Delta_n^p(\theta^c, \hat{f}^c) = f_n^c + \sum_{k=1}^p \gamma_k(\theta^c) f_{n-k}^c = 0, \quad n = N, N-1, \dots, N-p+1, \quad (1.2)$$

and

$$\Delta_n^p(\theta^s, \hat{f}^s) = f_n^s + \sum_{k=1}^p \gamma_k(\theta^s) f_{n-k}^s = 0, \quad n = N, N-1, \dots, N-p+1. \quad (1.3)$$

Then, θ^c and θ^s can be determined from (1.1), as the roots of the corresponding polynomials.

According to systems (1.2) and (1.3), coefficients $\gamma_k(\theta)$ would have the following asymptotic expansions (if f is enough smooth, see below)

$$\gamma_j(\theta^c) = \sum_{t=0}^{\infty} \frac{\gamma_{j,t}^c}{N^t}, \quad \gamma_j(\theta^s) = \sum_{t=0}^{\infty} \frac{\gamma_{j,t}^s}{N^t} \quad (1.4)$$

with some constants $\gamma_{j,t}^c$ and $\gamma_{j,t}^s$. In particular (see [27]),

$$\gamma_j(\theta^c) = O(1), \quad \gamma_j(\theta^s) = O(1), \quad N \rightarrow \infty. \quad (1.5)$$

More precisely,

$$\gamma_{j,0}^s = \gamma_{j,0}^c = \binom{p}{j}. \quad (1.6)$$

First, we introduce some lemmas.

Lemma 1.1 *Assume $f \in C^{2m+1}[-1, 1]$, $m \geq 0$ and $f^{(2m+1)} \in BV[-1, 1]$. Then, the following asymptotic expansions are valid*

$$f_n^c = (-1)^n \sum_{k=0}^m \frac{A_{2k+1}(f)}{(\pi n)^{2k+2}} + o(n^{-2m-2}), \quad n \rightarrow \infty, \quad (1.7)$$

and

$$f_n^s = (-1)^{n+1} \sum_{k=0}^m \frac{B_{2k+1}(f)}{(\pi(n - \frac{1}{2}))^{2k+2}} + o(n^{-2m-2}), \quad n \rightarrow \infty. \quad (1.8)$$

Lemma 1.2 *Assume $f \in C^{2m+2}[-1, 1]$, $m \geq 0$ and $f^{(2m+2)} \in BV[-1, 1]$. Then, the following asymptotic expansions are valid*

$$f_n^c = (-1)^n \sum_{k=0}^m \frac{A_{2k+1}(f)}{(\pi n)^{2k+2}} + o(n^{-2m-3}), \quad n \rightarrow \infty, \quad (1.9)$$

and

$$f_n^s = (-1)^{n+1} \sum_{k=0}^m \frac{B_{2k+1}(f)}{(\pi(n - \frac{1}{2}))^{2k+2}} + o(n^{-2m-3}), \quad n \rightarrow \infty. \quad (1.10)$$

Let

$$P_n(m) = \frac{(-1)^n}{(\pi n)^{2m+2}}, \quad Q_n(m) = \frac{(-1)^{n+1}}{(\pi(n - \frac{1}{2}))^{2m+2}}, \quad m \geq 0, n \geq 1. \quad (1.11)$$

and

$$\hat{P}(m) = \{P_n(m)\}_{n=1}^{\infty}, \quad \hat{Q}(m) = \{Q_n(m)\}_{n=1}^{\infty}. \quad (1.12)$$

Lemma 1.3 [10] *For each $p \geq 1$, the following estimates hold*

$$\Delta_n^p(\hat{P}(m)) = P_n(m) \frac{(-1)^p p!}{n^p} \binom{p+2m+1}{2m+1} + O(n^{-2m-p-3}), \quad (1.13)$$

and

$$\Delta_n^p(\hat{Q}(m)) = Q_n(m) \frac{(-1)^p p!}{n^p} \binom{p+2m+1}{2m+1} + O(n^{-2m-p-3}). \quad (1.14)$$

Proof. According to (0.29), we have

$$\begin{aligned} \Delta_n^p(\hat{P}(m)) &= \sum_{k=0}^p \binom{p}{k} P_{n-k}(m) \\ &= \sum_{k=0}^p \binom{p}{k} \frac{(-1)^{n+k}}{(\pi(n-k))^{2m+2}} \\ &= P_n(m) \sum_{j=0}^{\infty} \binom{j+2m+1}{2m+1} \frac{1}{n^j} \alpha_{p,j}, \end{aligned} \quad (1.15)$$

where

$$\alpha_{k,m} = \sum_{s=0}^k \binom{k}{s} (-1)^s s^m, \quad m \geq 0. \quad (1.16)$$

This concludes the proof as $\alpha_{p,j} = 0$, $j = 0, \dots, p-1$ and $\alpha_{p,p} = (-1)^p p!$ (see [22]). Similarly, we prove the second estimate. \square

Lemma 1.4 [10] *Assume $f \in C^{(2q+2p+2)}[-1, 1]$ and $f^{(2q+2p+2)} \in BV[-1, 1]$, $q \geq 0$, $p \geq 1$, and let the systems (1.2), (1.3) have unique solutions. If*

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1, \quad (1.17)$$

then, the following estimates hold

$$\Delta_n^w(\Delta_n^p(\theta^c, \hat{f}^c)) = O(n^{-w-2q-2}) + o(n^{-2q-2p-3}), \quad n \geq N+1, N \rightarrow \infty, \quad (1.18)$$

and

$$\Delta_n^w(\Delta_n^p(\theta^s, \hat{f}^s)) = O(n^{-w-2q-2}) + o(n^{-2q-2p-3}), \quad n \geq N+1, N \rightarrow \infty. \quad (1.19)$$

Proof. We provide the proof for f_n^c only. According to Lemma 1.2

$$f_n^c = (-1)^n \sum_{k=q}^{q+p} \frac{A_{2k+1}(f)}{(\pi n)^{2k+2}} + o(n^{-2q-2p-3}). \quad (1.20)$$

Then,

$$\begin{aligned} \Delta_n^w(\Delta_n^p(\theta, \hat{f}^c)) &= \sum_{s=0}^p \gamma_s(\theta^c) \Delta_{n-s}^w(\hat{f}^c) \\ &= \sum_{s=0}^p \gamma_s(\theta^c) \sum_{k=q}^{q+p} A_{2k+1}(f) \Delta_n^w(\hat{P}(k)) + o(n^{-2q-2p-3}). \end{aligned} \quad (1.21)$$

Taking into account estimates (1.5) and Lemma 1.3, we get the desired estimate. \square

Lemma 1.5 [10] *Assume $f \in C^{(2q+2p+2)}[-1, 1]$ and $f^{(2q+2p+2)} \in BV[-1, 1]$, $q \geq 0$, $p \geq 1$, and let systems (1.2), (1.3) have unique solutions. Let*

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1, \quad A_{2q+1}(f)B_{2q+1}(f) \neq 0. \quad (1.22)$$

Then, the following estimates are valid

$$\begin{aligned} \Delta_N^w(\Delta_n^p(\theta^c, \hat{f}^c)) &= A_{2q+1}(f) \frac{(-1)^{N+w}(2q+w+1)!}{\pi^{2q+2} N^{2q+p+w+2}(2q+1)!} \sum_{t=0}^p \beta_t^c(p-t) \binom{2q+w+t+1}{2q+w+1} \\ &\quad + O(N^{-2q-w-p-3}) + o(N^{-2q-2p-3}), \end{aligned} \quad (1.23)$$

and

$$\begin{aligned} \Delta_N^w(\Delta_n^p(\theta^s, \hat{f}^s)) &= B_{2q+1}(f) \frac{(-1)^{N+1+w}(2q+w+1)!}{\pi^{2q+2} N^{2q+p+w+2}(2q+1)!} \sum_{t=0}^p \beta_t^s(p-t) \binom{2q+w+t+1}{2q+w+1} \\ &\quad + O(N^{-2q-w-p-3}) + o(N^{-2q-2p-3}), \end{aligned} \quad (1.24)$$

where

$$\begin{aligned} \beta_u^c(t) &= \sum_{j=0}^p (-1)^j \gamma_{j,t}^c j^u, \\ \beta_u^s(t) &= \sum_{j=0}^p (-1)^j \gamma_{j,t}^s j^u, \end{aligned} \quad (1.25)$$

and $\gamma_{j,t}^c, \gamma_{j,t}^s$ are the coefficients of the following asymptotic expansions

$$\begin{aligned} \gamma_j(\theta^c) &= \sum_{t=0}^{2p+1} \frac{\gamma_{j,t}^c}{N^t} + o(N^{-2p-1}), \\ \gamma_j(\theta^s) &= \sum_{t=0}^{2p+1} \frac{\gamma_{j,t}^s}{N^t} + o(N^{-2p-1}). \end{aligned} \quad (1.26)$$

Proof. We prove (1.24). Estimate (1.23) can be handled similarly. The existence of asymptotic expansions (1.26) follows from the smoothness of f and the solutions of systems (1.2), (1.3). Then, we have

$$\begin{aligned}\Delta_N^w(\Delta_n^p(\theta^c, \hat{f}^c)) &= \sum_{s=0}^p \gamma_s(\theta^c) \Delta_{N-s}^w(\hat{f}^c) \\ &= \sum_{k=0}^w \binom{w}{k} \sum_{s=0}^p \gamma_s(\theta^c) f_{N-k-s}^c,\end{aligned}\tag{1.27}$$

where $\gamma_s(\theta^c)$ is the solution of system (1.2). From (1.20), we derive

$$\begin{aligned}f_{N-s-k}^c &= \frac{(-1)^{N-s-k}}{(\pi N)^{2q+2}} \sum_{j=0}^{2p+1} \frac{1}{N^j} \sum_{l=0}^{\lfloor \frac{j}{2} \rfloor} \frac{A_{2q+2l+1}(f)}{\pi^{2l}} \binom{2q+j+1}{2q+2l+1} (k+s)^{j-2l} \\ &\quad + o(N^{-2q-2p-3}).\end{aligned}\tag{1.28}$$

Substituting this and the first equation of (1.26) into (1.27), we obtain

$$\begin{aligned}\Delta_N^w(\Delta_n^p(\theta, \hat{f}^c)) &= \frac{(-1)^N}{(\pi N)^{2q+2}} \sum_{j=0}^{2p+1} \frac{1}{N^j} \sum_{t=0}^j \sum_{l=0}^{\lfloor \frac{j-t}{2} \rfloor} \frac{A_{2q+2l+1}(f)}{\pi^{2l}} \binom{2q+j-t+1}{2q+2l+1} \\ &\quad \times \sum_{u=0}^{j-t-2l} \binom{j-t-2l}{u} \alpha_{w,u} \beta_{j-t-2l-u}^c(t) \\ &\quad + o(N^{-2q-2p-3}),\end{aligned}\tag{1.29}$$

where $\alpha_{w,u}$ is defined by (1.16). Taking into account that (see [22]) $\alpha_{w,u} = 0$, $u = 0, \dots, w-1$, we get

$$\begin{aligned}\Delta_N^w(\Delta_n^p(\theta, \hat{f}^c)) &= \frac{(-1)^N}{\pi^{2q+2} N^{w+2q+2}} \sum_{j=0}^{2p-w+1} \frac{1}{N^j} \sum_{t=0}^j \sum_{l=0}^{\lfloor \frac{j-t}{2} \rfloor} \frac{A_{2q+2l+1}(f)}{\pi^{2l}} \binom{2q+t+w+1}{2q+2l+1} \\ &\quad \times \sum_{u=0}^{t-2l} \binom{t+w-2l}{u} \alpha_{w,t-2l-u+w} \beta_u^c(j-t) \\ &\quad + o(N^{-2q-2p-3}).\end{aligned}\tag{1.30}$$

From [27], we know that

$$\beta_u^c(j-t) = 0, \quad j = 0, \dots, p-1; \quad 0 \leq t \leq j; \quad 0 \leq u \leq t.\tag{1.31}$$

Thus, from (1.30), we obtain

$$\begin{aligned}\Delta_N^w(\Delta_n^p(\theta, \hat{f}^c)) &= \frac{(-1)^N}{\pi^{2q+2} N^{p+w+2q+2}} \sum_{t=0}^p \sum_{l=0}^{\lfloor \frac{t}{2} \rfloor} \frac{A_{2q+2l+1}(f)}{\pi^{2l}} \binom{2q+t+w+1}{2q+2l+1} \\ &\quad \times \sum_{u=0}^{t-2l} \binom{t+w-2l}{u} \alpha_{w,t-2l-u+w} \beta_u^c(p-t) \\ &\quad + O(N^{-p-w-2q-3}) + o(N^{-2q-2p-3}).\end{aligned}\tag{1.32}$$

It remains to notice that only the term $u = t, l = 0$ is not zero, and therefore,

$$\begin{aligned} \Delta_N^w(\Delta_n^p(\theta, \hat{f}^c)) &= A_{2q+1}(f) \frac{(-1)^N \alpha_{w,w}}{\pi^{2q+2} N^{p+w+2q+2}} \sum_{t=0}^p \binom{2q+t+w+1}{2q+1} \binom{t+w}{t} \beta_t^c(p-t) \\ &\quad + O(N^{-p-w-2q-3}) + o(N^{-2q-2p-3}). \end{aligned} \quad (1.33)$$

This concludes the proof since $\alpha_{w,w} = (-1)^w w!$. \square

Theorem 1.1 [10] *Assume $f \in C^{(2q+2p+2)}[-1, 1]$ and $f^{(2q+2p+2)} \in BV[-1, 1]$, $q \geq 0, p \geq 1$, and let systems (1.2), (1.3) have unique solutions. If*

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1, \quad A_{2q+1}(f)B_{2q+1}(f) \neq 0, \quad (1.34)$$

then, the following estimates are valid for $x \in (-1, 1)$

$$\begin{aligned} R_{N,p}^{\cos}(f, \theta^c, x) &= A_{2q+1}(f) \frac{(-1)^{N+1} (2q+p+1)! p!}{2^{2p+1} \pi^{2q+2} N^{2q+2p+2} (2q+1)!} \frac{\cos \frac{\pi x}{2} (2N-2p+1)}{\cos^{2p+1} \frac{\pi x}{2}} \\ &\quad + o(N^{-2q-2p-2}), \end{aligned} \quad (1.35)$$

and

$$\begin{aligned} R_{N,p}^{\sin}(f, \theta^s, x) &= B_{2q+1}(f) \frac{(-1)^N (2q+p+1)! p!}{2^{2p+1} \pi^{2q+2} N^{2q+2p+2} (2q+1)!} \frac{\sin \frac{\pi x}{2} (2N-2p)}{\cos^{2p+1} \frac{\pi x}{2}} \\ &\quad + o(N^{-2q-2p-2}). \end{aligned} \quad (1.36)$$

Proof. We estimate $R_{N,p}^{\cos}(f, \theta^c, x)$. According to (1.6), we have

$$\prod_{k=1}^p (1 + \theta_k^c e^{i\pi k x}) \rightarrow (1 + e^{i\pi x})^p, \quad N \rightarrow \infty, \quad (1.37)$$

and it remains to estimate only the sum in the right hand side of (0.36)

$$\begin{aligned} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta^c, \hat{f}^c) e^{i\pi n x} &= -e^{i\pi(N+1)x} \sum_{w=0}^{2p+1} \frac{\Delta_N^w(\Delta_n^p(\theta^c, \hat{f}^c))}{(1 + e^{i\pi x})^{w+1}} \\ &\quad + \frac{1}{(1 + e^{i\pi x})^{2p+2}} \sum_{n=N+1}^{\infty} \Delta_n^{2p+2}(\Delta_n^p(\theta^c, \hat{f}^c)) e^{i\pi n x}. \end{aligned} \quad (1.38)$$

Taking into account that

$$\Delta_N^k(\Delta_n^p(\theta^c, \hat{f}^c)) = \sum_{s=0}^k \binom{k}{s} \Delta_{N-s}^p(\theta^c, \hat{f}^c), \quad (1.39)$$

we see from (0.43) that

$$\Delta_N^k(\Delta_n^p(\theta^c, \hat{f}^c)) = 0, \quad k = 0, \dots, p-1. \quad (1.40)$$

Therefore,

$$\begin{aligned}
\sum_{n=N+1}^{\infty} \Delta_n^p(\theta^c, \hat{f}^c) e^{i\pi n x} &= -e^{i\pi(N+1)x} \frac{\Delta_N^p(\Delta_n^p(\theta^c, \hat{f}^c))}{(1 + e^{i\pi x})^{p+1}} \\
&\quad - e^{i\pi(N+1)x} \sum_{w=p+1}^{2p+1} \frac{\Delta_N^w(\Delta_n^p(\theta^c, \hat{f}^c))}{(1 + e^{i\pi x})^{w+1}} \\
&\quad + \frac{1}{(1 + e^{i\pi x})^{2p+2}} \sum_{n=N+1}^{\infty} \Delta_n^{2p+2}(\Delta_n^p(\theta^c, \hat{f}^c)) e^{i\pi n x}.
\end{aligned} \tag{1.41}$$

Lemma 1.4 shows that

$$\Delta_n^{2p+2}(\Delta_n^p(\theta^c, \hat{f}^c)) = o(n^{-2q-2p-3}), \quad n \rightarrow \infty. \tag{1.42}$$

Hence, the last term in the right hand side of (1.41) is $o(N^{-2p-2q-2})$. According to Lemma 1.5

$$\Delta_N^w(\Delta_n^p(\theta^c, \hat{f}^c)) = O(N^{-2q-w-p-2}) + o(N^{-2q-w-p-2}), \quad N \rightarrow \infty. \tag{1.43}$$

As parameter w is ranges from $w = p + 1$ to $w = 2p + 1$, then the latest is $O(N^{-2q-2p-3})$.

Hence,

$$\begin{aligned}
\sum_{n=N+1}^{\infty} \Delta_n^p(\theta^c, \hat{f}^c) e^{i\pi n x} &= -e^{i\pi(N+1)x} \frac{\Delta_N^p(\Delta_n^p(\theta^c, \hat{f}^c))}{(1 + e^{i\pi x})^{2p+1}} \\
&\quad + o(N^{-2q-2p-2}).
\end{aligned} \tag{1.44}$$

Similarly,

$$\begin{aligned}
\sum_{n=N+1}^{\infty} \Delta_n^p(\theta^c, \hat{f}^c) e^{-i\pi n x} &= -e^{-i\pi(N+1)x} \frac{\Delta_N^p(\Delta_n^p(\theta^c, \hat{f}^c))}{(1 + e^{-i\pi x})^{2p+1}} \\
&\quad + o(N^{-2q-2p-2}).
\end{aligned} \tag{1.45}$$

Therefore,

$$\begin{aligned}
R_{N,p}^{\cos}(f, \theta^c, x) &= -e^{i\pi(N+1)x} \frac{\Delta_N^p(\Delta_n^p(\theta^c, \hat{f}^c))}{(1 + e^{i\pi x})^{2p+1}} \\
&\quad - e^{-i\pi(N+1)x} \frac{\Delta_N^p(\Delta_n^p(\theta^c, \hat{f}^c))}{(1 + e^{-i\pi x})^{2p+1}} \\
&\quad + o(N^{-2q-2p-2}).
\end{aligned} \tag{1.46}$$

Finally, we need to estimate $\Delta_N^p(\Delta_n^p(\theta^c, \hat{f}^c))$. Again by Lemma 1.5, we have

$$\begin{aligned}
\Delta_N^p(\Delta_n^p(\theta^c, \hat{f}^c)) &= A_{2q+1}(f) \frac{(-1)^{N+p}}{N^{2p+2q+2} \pi^{2q+2} (2q+1)!} \\
&\quad \times \sum_{t=0}^p \beta_t^c(p-t) \frac{(2q+p+t+1)!}{t!} \\
&\quad + o(N^{-2q-2p-2}).
\end{aligned} \tag{1.47}$$

It is possible to show (details see in [27]) that the sum in the right-hand side of (1.47) equals to $(-1)^p p!(p+2q+1)!$. Hence,

$$\begin{aligned} \Delta_N^p(\Delta_n^p(\theta^c, f^c)) &= A_{2q+1}(f) \frac{(-1)^N (2q+1+p)! p!}{N^{2p+2q+2} \pi^{2q+2} (2q+1)!} \\ &\quad + o(N^{-2q-2p-2}). \end{aligned} \tag{1.48}$$

Together with (1.46), it implies

$$\begin{aligned} R_{N,p}^{\cos}(f, \theta^c, x) &= A_{2q+1}(f) \frac{(-1)^{N+1} (2q+1+p)! p!}{N^{2p+2q+2} \pi^{2q+2} (2q+1)!} \operatorname{Re} \left[\frac{e^{i\pi(N+1)x}}{(1+e^{i\pi x})^{2p+1}} \right] \\ &\quad + o(N^{-2q-2p-2}). \end{aligned} \tag{1.49}$$

Similarly, we can show that

$$\begin{aligned} R_{N,p}^{\sin}(f, \theta^s, x) &= B_{2q+1}(f) \frac{(-1)^N (2q+1+p)! p!}{N^{2p+2q+2} \pi^{2q+2} (2q+1)!} \operatorname{Re} \left[\frac{e^{i\pi(N+\frac{1}{2})x}}{i(1+e^{i\pi x})^{2p+1}} \right] \\ &\quad + o(N^{-2q-2p-2}), \end{aligned} \tag{1.50}$$

which completes the proof. \square

Note that for $p=0$, Theorem 1.1 coincides with Theorem 0.4.

Let

$$f_q(x) = \sin(x-1)(x^2-1)^{2q}, \quad q = 0, 1, 2, \dots \tag{1.51}$$

It is easy to verify that these functions obey the first q derivative conditions.

Figure 1.1 shows the results of the approximation of $f_1(x)$ by the modified Fourier expansion ($p=0$) and the MFP-approximations ($p=1, 2, 3$). We see a tremendous increase in accuracy while applying the MFP-approximations for smooth function on $[-0.7, 0.7]$. For example, in case of $p=3$, the improvement is $3.8 \cdot 10^7$ times.

Next, we investigate the pointwise convergence of the MFP-approximations at the endpoints $x = \pm 1$. We estimate (0.36) and (0.37) for $x = \pm 1$.

Taking into account (1.6), we see that $\theta_k^c, \theta_k^s \rightarrow 1$, as $N \rightarrow \infty$. Let

$$\begin{aligned} \theta_k^c &= 1 - \frac{\tau_k^c}{N} + o(N^{-1}), \quad k = 1, \dots, p, \\ \theta_k^s &= 1 - \frac{\tau_k^s}{N} + o(N^{-1}), \quad k = 1, \dots, p. \end{aligned} \tag{1.52}$$

To find τ^c and τ^s , we compare two results that outline the behavior of $\Delta_n^p(\theta^c, \hat{f}^c)$ and $\Delta_n^p(\theta^s, \hat{f}^s)$.

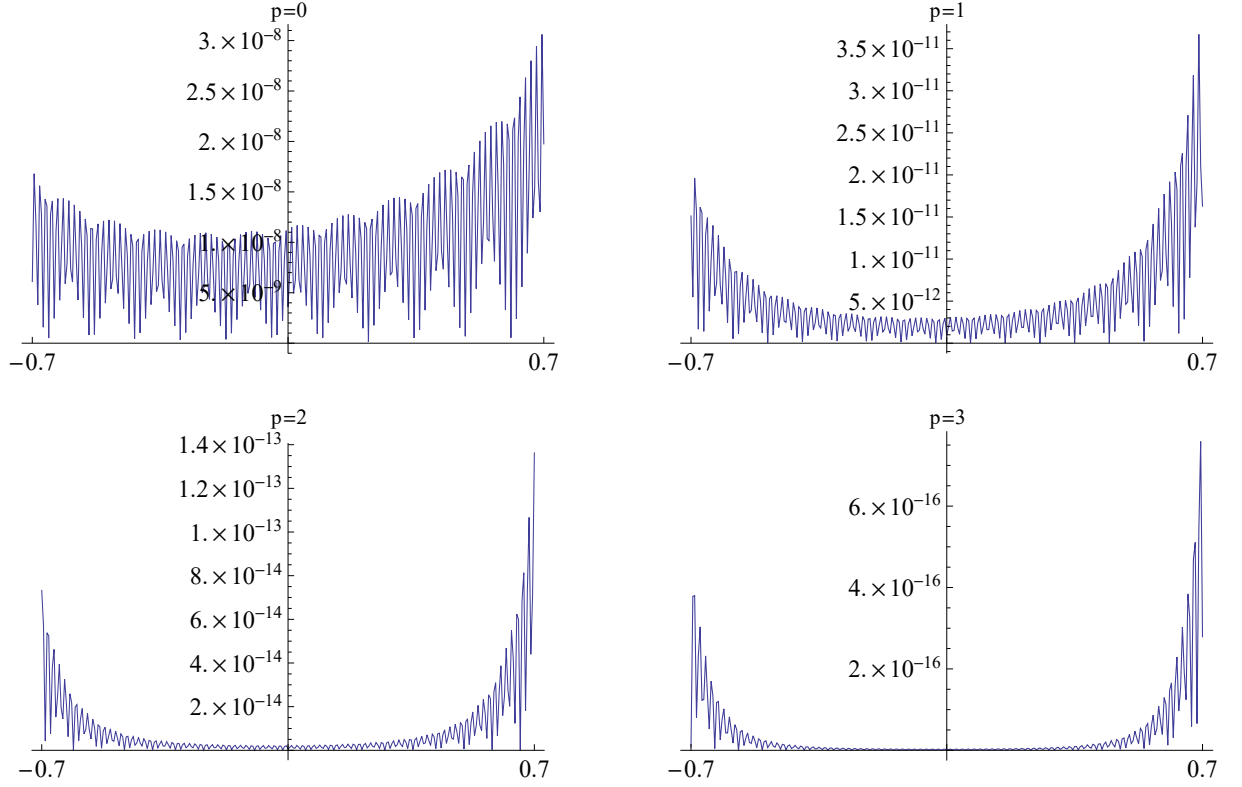


Figure 1.1: The graphs of $|R_{N,p}(f, \theta^c, \theta^s, x)|$ on $[-0.7, 0.7]$ for $N = 64$ while approximating $f_1(x)$ (see (1.51)) by the modified Fourier expansions ($p = 0$) and the MFP-approximations ($p = 1, 2, 3$).

Lemma 1.6 [10] *Let $f \in C^{(2q+p+1)}[-1, 1]$, $f^{(2q+p+1)} \in BV[-1, 1]$, $q \geq 0$, and*

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1, \quad A_{2q+1}(f)B_{2q+1}(f) \neq 0. \quad (1.53)$$

Let systems (1.2), (1.3) have unique solutions θ^c and θ^s , respectively. Then, the following estimates hold as $N \rightarrow \infty$ and $n > N$

$$\begin{aligned} \Delta_n^p(\theta^c, \hat{f}^c) &= A_{2q+1}(f) \frac{(-1)^n (2q+p+1)!}{N^p n^{2q+2} \pi^{2q+2} (2q+1)!} \left(1 - \frac{N}{n}\right)^p \\ &\quad + o(N^{-p}) \frac{1}{n^{2q+2}}, \end{aligned} \quad (1.54)$$

and

$$\begin{aligned} \Delta_n^p(\theta^s, \hat{f}^s) &= B_{2q+1}(f) \frac{(-1)^{n+1} (2q+p+1)!}{N^p n^{2q+2} \pi^{2q+2} (2q+1)!} \left(1 - \frac{N}{n - \frac{1}{2}}\right)^p \\ &\quad + o(N^{-p}) \frac{1}{n^{2q+2}}. \end{aligned} \quad (1.55)$$

Proof. We prove only the first estimate. The proof, in general, imitate the one of Lemma 1.5,

so we omit some details. Let $\gamma_{s,t}^c$ be the coefficients of the asymptotic expansion

$$\gamma_s(\theta^c) = \sum_{t=0}^p \frac{\gamma_{s,t}^c}{N^t} + o(N^{-p}). \quad (1.56)$$

We replicate the arguments in the proof of Lemma 1.5, then apply Lemma 1.1 (when p is odd) or Lemma 1.2 (when p is even), and obtain

$$\begin{aligned} \Delta_n^p(\theta^c, \hat{f}^c) &= \frac{(-1)^n}{(\pi n)^{2q+2}} \sum_{j=0}^p \frac{1}{N^j} \sum_{t=0}^j \frac{1}{\left(\frac{n}{N}\right)^t} \sum_{l=0}^{\lfloor \frac{t}{2} \rfloor} \frac{A_{2q+2l+1}(f)}{\pi^{2l}} \binom{2q+2l+1}{2q+t+1} \beta_{t-2l}^c(j-t) \\ &\quad + o(N^{-p}) \frac{1}{n^{2q+2}}, \end{aligned} \quad (1.57)$$

where $\beta_u^c(j-t)$ are defined by (1.25). From the proof of Lemma 1.5, we know that

$$\beta_u^c(j-t) = 0, \quad j = 0, \dots, p-1; \quad 0 \leq p \leq j; \quad 0 \leq u \leq t. \quad (1.58)$$

Therefore,

$$\begin{aligned} \Delta_n^p(\theta^c, \hat{f}^c) &= A_{2q+1}(f) \frac{(-1)^n}{N^p (\pi n)^{2q+2}} \sum_{t=0}^p \frac{1}{\left(\frac{n}{N}\right)^t} \binom{2q+t+1}{2q+1} \beta_t^c(p-t) \\ &\quad + o(N^{-p}) \frac{1}{n^{2q+2}}. \end{aligned} \quad (1.59)$$

This concludes the proof (see [27]). \square

We omit the proof of the next Lemma, as Lemma 2.1 proves a more general result.

Lemma 1.7 [10, 12] *Let $f^{(2q+p+1)} \in AC[-1, 1]$, $q \geq 0$. Let*

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1, \quad A_{2q+1}(f) B_{2q+1}(f) \neq 0, \quad (1.60)$$

and

$$\theta_k^c = 1 - \frac{\tau_k^c}{N}, \quad \theta_k^s = 1 - \frac{\tau_k^s}{N}, \quad k = 1, \dots, p. \quad (1.61)$$

Then, the following estimates hold for $n > N$, as $N \rightarrow \infty$

$$\begin{aligned} \Delta_n^p(\theta_k^c, \hat{f}^c) &= A_{2q+1}(f) \frac{(-1)^{n+p}}{n^{2q+2} (2q+1)! \pi^{2q+2}} \sum_{k=0}^p \frac{(2q+p-k+1)! (-1)^k \gamma_k(\tau^c)}{N^k n^{p-k}} \\ &\quad + o(N^{-p}) \frac{1}{n^{2q+2}}, \end{aligned} \quad (1.62)$$

and

$$\begin{aligned} \Delta_n^p(\theta_k^s, \hat{f}_n^s) &= B_{2q+1}(f) \frac{(-1)^{n+p+1}}{n^{2q+2} (2q+1)! \pi^{2q+2}} \sum_{k=0}^p \frac{(2q+p-k+1)! (-1)^k \gamma_k(\tau^s)}{N^k (n - \frac{1}{2})^{p-k}} \\ &\quad + o(N^{-p}) \frac{1}{n^{2q+2}}, \end{aligned} \quad (1.63)$$

where

$$\prod_{k=1}^p (1 + \tau_k^c x) = \sum_{k=0}^p \gamma_k(\tau^c) x^k, \quad (1.64)$$

and

$$\prod_{k=1}^p (1 + \tau_k^s x) = \sum_{k=0}^p \gamma_k(\tau^s) x^k. \quad (1.65)$$

Comparing Lemmas 1.6 and 1.7, we get

$$\gamma_k(\tau^c) = \gamma_k(\tau^s) = \frac{(2q + p + 1)!}{(2q + p - k + 1)!} \binom{p}{k}. \quad (1.66)$$

Now, recall ([34]) that the generalized Laguerre polynomials $L_p^{(\alpha)}(x)$ have the following closed form

$$L_p^{(\alpha)}(x) = \sum_{k=0}^p (-1)^k \frac{(p + \alpha)!}{k!(p - k)!(\alpha + k)!} x^k. \quad (1.67)$$

Also, it is worth to noting that the generalized Laguerre polynomial $L_p^{(\alpha)}(x)$ has p real-valued and strictly positive simple roots.

From (1.66) and (1.67), it follows that $\tau_k^c = \tau_k^s = \tau_k$, $k = 1, \dots, p$ are the roots of the generalized Laguerre polynomial $L_p^{(2q+1)}(x)$, which leads to the following theorem.

Theorem 1.2 [11] *Assume $f \in C^{2q+p+2}[-1, 1]$, $q \geq 0$, $p \geq 0$ and $f^{(2q+p+2)} \in BV[-1, 1]$. Let*

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q - 1, \quad (1.68)$$

and

$$A_{2q+1}(f)B_{2q+1}(f) \neq 0. \quad (1.69)$$

Let systems (1.2), (1.3) have unique solutions θ^c and θ^s , respectively. Then, the following estimate holds

$$R_{N,p}(f, \theta^c, \theta^s, \pm 1) = \frac{h_{p,q}}{\pi^{2q+2}(2q+1)N^{2q+1}} (A_{2q+1}(f) \pm B_{2q+1}(f)) + o(N^{-2q-1}), \quad (1.70)$$

where

$$h_{p,q} = \frac{p!(2q+1)!}{(2q+p+1)!}. \quad (1.71)$$

Proof. As we mentioned above parameters θ^c and θ^s have asymptotic expansions as in (1.52), where parameters $\tau_k^c = \tau_k^s = \tau_k$, $k = 1, \dots, p$ and τ_k are the roots of the generalized Laguerre polynomial

$$L_p^{(2q+1)}(x) = \sum_{k=0}^p (-1)^k \frac{(p+2q+1)!}{k!(p-k)!(2q+1+k)!} x^k. \quad (1.72)$$

From here and the Vieta's formula, we also have

$$\prod_{k=1}^p \tau_k = \frac{(2q+1+p)!}{(2q+1)!}. \quad (1.73)$$

In view of (0.36) and (0.37), we write

$$R_{N,p}^{cos}(f, \theta^c, \pm 1) = \frac{N^p}{\prod_{k=1}^p (\tau_k + o(1))} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta^c, \hat{f}^c) (-1)^n, \quad (1.74)$$

and

$$R_{N,p}^{sin}(f, \theta^s, \pm 1) = \pm \frac{N^p}{\prod_{k=1}^p (\tau_k + o(1))} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta^s, \hat{f}^s) (-1)^n. \quad (1.75)$$

From (1.66) and Lemma 1.6, we get

$$R_{N,p}^{cos}(f, \theta^c, \pm 1) = \frac{A_{2q+1}(f)}{\pi^{2q+2}} \frac{(2q+p+1)!}{(2q+1)! \prod_{k=1}^p \tau_k} \sum_{n=N+1}^{\infty} \frac{1}{n^{2q+2}} \left(1 - \frac{N}{n}\right)^p + o(N^{-2q-1}). \quad (1.76)$$

Finally,

$$R_{N,p}^{cos}(f, \theta^c, \theta^s, \pm 1) = \frac{A_{2q+1}(f)}{\pi^{2q+2} N^{2q+1}} \frac{h_{p,q}}{(2q+1)} + o(N^{-2q-1}), \quad (1.77)$$

where

$$h_{p,q} = (2q+1) \sum_{k=0}^p \binom{p}{k} \frac{(-1)^k}{2q+k+1}. \quad (1.78)$$

Similarly, we show that

$$R_{N,p}^{sin}(f, \theta^c, \theta^s, \pm 1) = \pm \frac{B_{2q+1}(f)}{\pi^{2q+2} N^{2q+1}} \frac{h_{p,q}}{(2q+1)} + o(N^{-2q-1}), \quad (1.79)$$

which concludes the proof (see [35]). \square

Note that, for $p = 0$, Theorem 1.2 coincides with Theorem 0.5. Comparison of Theorems 1.2 and 0.5 shows that the expansions by the modified Fourier system and the MFP-approximations have the same convergence rates at the endpoints $x = \pm 1$. However, the comparison of

$p \backslash q$	q=0	q=1	q=2	q=3	q=4	q=5	q=6
$p = 1$	2	4	6	8	10	12	14
$p = 2$	3	10	21	36	55	78	105
$p = 3$	4	20	56	120	220	364	560
$p = 4$	5	35	126	330	715	1365	2380

Table 1.1: The values of $1/h_{p,q}$.

constants $h_{p,q}$ with $h_{0,q} = 1$ shows that the MFP-approximations are much more accurate than the classical expansions and asymptotic improvement is by factor $h_{0,q}/h_{p,q} = 1/h_{p,q}$. Table 1.1 presents the numerical values of the ratio $1/h_{p,q}$.

It would be interesting to compare the asymptotic improvement with actual improvement for moderate values of N . Table 1.2 shows the values of $\frac{\max_{x=\pm 1} |R_N(f, x)|}{\max_{x=\pm 1} |R_{N,p}(f, \theta^c, \theta^s, x)|}$ for $f_q(x)$ (see (1.51)) for $N = 64$.

We see that for small values of p and q , the corresponding numbers of Tables 1.1 and 1.2 are rather close.

$p \backslash q$	q=0	q=1	q=2	q=3	q=4	q=5	q=6
$p = 1$	1.97	3.82	5.55	7.17	8.70	10.13	11.49
$p = 2$	2.91	9.10	17.93	28.90	41.54	55.50	70.50
$p = 3$	3.81	17.33	44.11	86.08	143.93	217.63	306.81
$p = 4$	4.69	28.88	91.42	211.17	404.31	684.00	1060.94

Table 1.2: The values of ratio $\frac{\max_{x=\pm 1} |R_N(f, x)|}{\max_{x=\pm 1} |R_{N,p}(f, \theta^c, \theta^s, x)|}$ while approximating (1.51) for $N = 64$.

Figure 1.2 demonstrates the values of $-\log_{10} \left(\max_{x=\pm 1} |R_{N,p}(f, \theta^c, \theta^s, x)| \right)$ for different values of N and $p = 0, 1, 2, 3$ while approximating $f_1(x)$ (see (1.51)). The case $p = 0$ corresponds to the expansions by the modified Fourier system.

Now, we investigate the L_2 -convergence of the modified Fourier-Pade approximations.

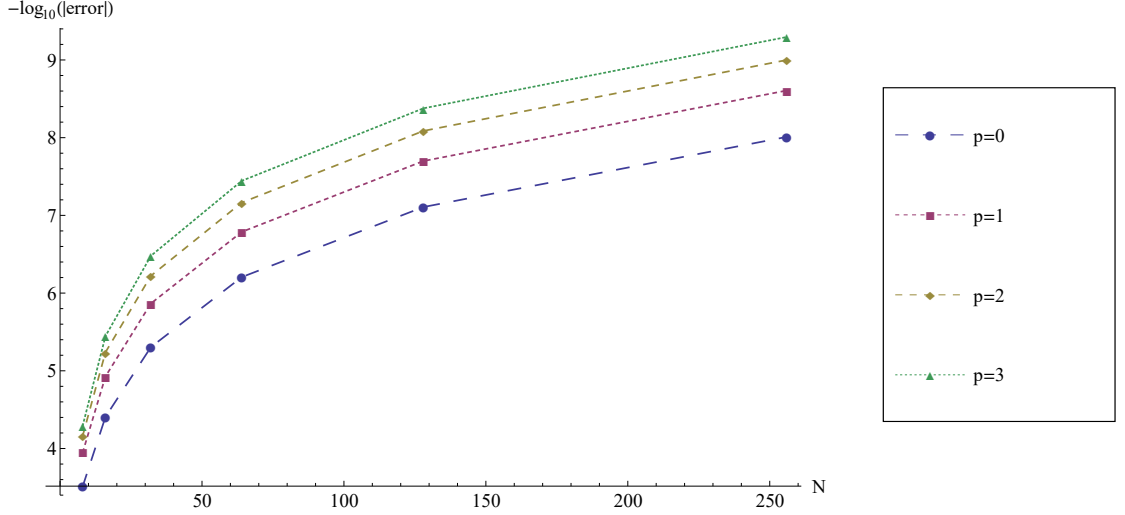


Figure 1.2: The values of $-\log_{10} \left(\max_{x=\pm 1} |R_{N,p}(f, \theta^c, \theta^s, x)| \right)$ for different N and p while approximating $f_1(x)$. The case $p = 0$ corresponds to the expansion by the modified Fourier system.

Theorem 1.3 [10] Let $f \in C^{2q+p+1}[-1, 1]$, $f^{(2q+p+1)} \in BV[-1, 1]$, $q \geq 0, p \geq 1$. Let θ^c, θ^s be the unique solutions of (1.2) and (1.3), respectively. If

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1, \quad A_{2q+1}(f)B_{2q+1}(f) \neq 0, \quad (1.80)$$

then, the following estimate holds

$$\lim_{N \rightarrow \infty} N^{2q+\frac{3}{2}} \|R_{N,p}\|_{L_2} = \frac{c_{p,q}}{\pi^{2q+2} \sqrt{4q+3}} \sqrt{A_{2q+1}^2(f) + B_{2q+1}^2(f)}, \quad (1.81)$$

where

$$c_{p,q} = \frac{(p+2q+1)! \sqrt{4q+3}}{(2q+1)!} \left(\int_1^\infty dt \left| \int_1^t \frac{(1-x)^p}{x^{2q+p+2}} \sum_{j=1}^p \frac{e^{-\tau_j(t-x)}}{\prod_{\substack{k=1 \\ k \neq j}}^p (\tau_j - \tau_k)} dx \right|^2 \right)^{\frac{1}{2}}, \quad (1.82)$$

and $\tau_k, k = 1, \dots, p$ are the roots of the generalized Laguerre polynomial $L_p^{(2q+1)}(x)$.

Proof. We have (details see in [10])

$$\|R_{N,p}(f, \theta, x)\|_{L_2}^2 = S_1 + S_2, \quad (1.83)$$

where

$$S_1 = \sum_{j=N+1}^{\infty} \left| \sum_{n=N+1}^j (-1)^n \Delta_n^p(\theta^c, \hat{f}^c) h_{j-n}^{\cos} \right|^2, \quad (1.84)$$

and

$$S_2 = \sum_{j=N+1}^{\infty} \left| \sum_{n=N+1}^j (-1)^n \Delta_n^p(\theta^s, \hat{f}^s) h_{j-n}^{sin} \right|^2, \quad (1.85)$$

where

$$h_k^{cos} = \sum_{i=1}^p \frac{(\theta_i^c)^{p+k-1}}{\prod_{j=1, j \neq i}^p (\theta_i^c - \theta_j^c)}, \quad (1.86)$$

and

$$h_k^{sin} = \sum_{i=1}^p \frac{(\theta_i^s)^{p+k-1}}{\prod_{j=1, j \neq i}^p (\theta_i^s - \theta_j^s)}. \quad (1.87)$$

Then,

$$\begin{aligned} h_{s-k}^{cos} &= \sum_{i=1}^p \frac{(\theta_i^c)^{p+s-n-1}}{\prod_{j=1, j \neq i}^p (\theta_i^c - \theta_j^c)} \\ &= N^{p-1} \sum_{i=1}^p \frac{(1 - \frac{\tau_i}{N} + o(N^{-1}))^{p+s-n-1}}{\prod_{j=1, j \neq i}^p (\tau_j - \tau_i + o(1))}. \end{aligned} \quad (1.88)$$

Similar estimate, we can write for h_{s-k}^{sin} .

Now, from estimates (1.54) and (1.55), we derive the limits

$$\lim_{N \rightarrow \infty} N^{4q+3} S_1 = c_{p,q}^2 A_{2q+1}^2,$$

and

$$\lim_{N \rightarrow \infty} N^{4q+3} S_2 = c_{p,q}^2 B_{2q+1}^2,$$

which conclude the proof. \square

Estimate (1.81) is valid also for $p = 0$, which coincides with Theorem 0.3 as $c_{0,q} = 1$. Comparison of Theorems 1.3 and 0.3 shows that the classical expansions and the MFP-approximations have the same convergence rates in the L_2 -norm. However, comparison of constants $c_{p,q}$ and $c_{0,q} = 1$ shows that the rational approximations are asymptotically more accurate and, the improvement is thanks to the factor $c_{0,q}/c_{p,q} = 1/c_{p,q}$. Table 1.3 shows the numerical values of ratio $1/c_{p,q}$. For example, when $q = 6$ and $p = 4$, the asymptotic improvement is 5595 times.

Let us see, how those asymptotic estimates could be achieved for moderate values of N for a specific function. Table 1.4 shows the values of $\frac{\|R_N\|_{L_2}}{\|R_{N,p}\|_{L_2}}$ while approximating $f_q(x)$ (see (1.51)) for $N = 64$. We see that the corresponding numbers in Tables 1.3 and 1.4 are close for small p and q .

$p \backslash q$	q=0	q=1	q=2	q=3	q=4	q=5	q=6
$p = 1$	3.4	6.6	9.9	13	16	19	22
$p = 2$	6.3	20	41	70	107	151	203
$p = 3$	9.8	46	125	265	481	791	1212
$p = 4$	13	89	310	797	1706	3229	5595

Table 1.3: The values of the ratio $1/c_{p,q}$.

$p \backslash q$	q=0	q=1	q=2	q=3	q=4	q=5	q=6
$p = 1$	3.3	6.3	9.1	11	15	16	18
$p = 2$	6.1	18	34	55	89	106	134
$p = 3$	9.2	38	96	185	343	461	647
$p = 4$	12	71	217	493	1042	1565	2413

Table 1.4: The values of $\frac{\|R_N\|_{L_2}}{\|R_{N,p}\|_{L_2}}$ while approximating (1.51) with $N = 64$.

2. Convergence of the Optimal MTR-approximations

on $|x| < 1$

Throughout this and the next section, we assume that parameters θ_k , $k = 1, \dots, p$ are defined by (see (0.45))

$$\theta_k = 1 - \frac{\tau_k}{N}, \quad \tau_k \neq 0, \quad k = 1, \dots, p. \quad (2.1)$$

Let $\tau = \{\tau_1, \dots, \tau_p\}$ and coefficients $\gamma_k(\tau)$ be defined by the following identity

$$\prod_{k=1}^p (1 + \tau_k x) = \sum_{k=0}^p \gamma_k(\tau) x^k. \quad (2.2)$$

Next lemma unveils the asymptotic expansions of $\Delta^w(\hat{\Delta}^p(\theta, \hat{f}^c))$ and $\Delta^w(\hat{\Delta}^p(\theta, \hat{f}^s))$, where

$$\hat{\Delta}^p(\theta, \hat{f}) = \{\Delta_n^p(\theta, \hat{f})\}_n. \quad (2.3)$$

Lemma 2.1 [12] Assume $f \in C^{2q+p+r+1}[-1, 1]$, $f^{(2q+p+r+1)} \in BV[-1, 1]$, $q \geq 0$, $r \geq 0$, $p \geq 1$,

and

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1. \quad (2.4)$$

Let θ_k , $k = 1, \dots, p$ be defined by (2.1). Then, the following estimates hold for $n > N$ as $N \rightarrow \infty$

$$\begin{aligned} \Delta_n^w(\hat{\Delta}^p(\theta, \hat{f}^c)) &= (-1)^n \sum_{k=0}^p \frac{\gamma_k(\tau)}{N^k} \sum_{t=2q+w}^{2q+r} \frac{(t+p-k+1)!}{n^{t+p-k+2}} \sum_{s=q}^{\lfloor \frac{t-w}{2} \rfloor} \frac{A_{2s+1}(f)}{\pi^{2s+2}(2s+1)!} \beta_{k,s,t}(w) \\ &\quad + \frac{o(N^{-p})}{n^{2q+r+2}}, \end{aligned} \quad (2.5)$$

$$\begin{aligned} \Delta_n^w(\hat{\Delta}^p(\theta, \hat{f}^s)) &= (-1)^{n+1} \sum_{k=0}^p \frac{\gamma_k(\tau)}{N^k} \sum_{t=2q+w}^{2q+r} \frac{(t+p-k+1)!}{n^{t+p-k+2}} \sum_{s=q}^{\lfloor \frac{t-w}{2} \rfloor} \frac{B_{2s+1}(f)}{\pi^{2s+2}(2s+1)!} \tilde{\beta}_{k,s,t}(w) \\ &\quad + \frac{o(N^{-p})}{n^{2q+r+2}}, \end{aligned} \quad (2.6)$$

where

$$\beta_{k,s,t}(w) = \sum_{\ell=w}^{t-2s} k^{t-2s-\ell} \frac{\alpha_{w+p-k, \ell+p-k}}{(t-2s-\ell)!(p-k+\ell)!}, \quad (2.7)$$

$$\tilde{\beta}_{k,s,t}(w) = \sum_{\ell=w}^{t-2s} \left(k + \frac{1}{2}\right)^{t-2s-\ell} \frac{\alpha_{w+p-k, \ell+p-k}}{(t-2s-\ell)!(p-k+\ell)!}, \quad (2.8)$$

with

$$\alpha_{k,j} = \sum_{s=0}^k \binom{k}{s} (-1)^s s^j, \quad j \geq 0. \quad (2.9)$$

Proof. In view of (0.28) and (2.1), we have

$$\Delta_n^p(\theta, \hat{f}^c) = \sum_{k=0}^p \frac{(-1)^k \gamma_k(\tau)}{N^k} \Delta_{n-k}^{p-k}(\hat{f}^c). \quad (2.10)$$

Taking into account that $\Delta_n^w(\hat{\Delta}^{p-k}(\hat{f}^c)) = \Delta_n^{w+p-k}(\hat{f}^c)$, and using (0.29), we get

$$\Delta_n^w(\hat{\Delta}^p(\theta, \hat{f}^c)) = \sum_{k=0}^p \frac{(-1)^k \gamma_k(\tau)}{N^k} \sum_{j=0}^{w+p-k} \binom{w+p-k}{j} f_{n-(k+j)}^c. \quad (2.11)$$

Application of Lemma 1.1, when $p+r$ is odd, and Lemma 1.2, when $p+r$ is even, leads to the following asymptotic expansion ($h = p - k + r$)

$$\begin{aligned} \Delta_n^w(\hat{\Delta}^p(\theta, \hat{f}^c)) &= (-1)^n \sum_{k=0}^p \frac{\gamma_k(\tau)}{N^k} \sum_{j=0}^{w+p-k} (-1)^j \binom{w+p-k}{j} \\ &\quad \times \sum_{s=q}^{q+\lfloor \frac{h}{2} \rfloor} \frac{A_{2s+1}(f)}{(\pi n)^{2s+2}} \frac{1}{\left(1 - \frac{j+k}{n}\right)^{2s+2}} \\ &\quad + \frac{o(N^{-p})}{n^{2q+r+2}}. \end{aligned} \quad (2.12)$$

Then,

$$\begin{aligned}
\Delta_n^w(\hat{\Delta}^p(\theta, \hat{f}^c)) &= (-1)^n \sum_{k=0}^p \frac{\gamma_k(\tau)}{N^k} \sum_{j=0}^{w+p-k} (-1)^j \binom{w+p-k}{j} \\
&\times \sum_{s=q}^{q+\lfloor \frac{t}{2} \rfloor} \frac{A_{2s+1}(f)}{(\pi n)^{2s+2}} \sum_{t=2s+1}^{\infty} \binom{t}{2s+1} \frac{(k+j)^{t-2s-1}}{n^{t-2s-1}} \\
&+ \frac{o(N^{-p})}{n^{2q+r+2}}.
\end{aligned} \tag{2.13}$$

Finally,

$$\begin{aligned}
\Delta_n^w(\hat{\Delta}^p(\theta, \hat{f}^c)) &= (-1)^n \sum_{k=0}^p \frac{\gamma_k(\tau)}{N^k} \sum_{t=2q}^{2q+h} \frac{1}{n^{t+2}} \sum_{s=q}^{\lfloor \frac{t}{2} \rfloor} \frac{A_{2s+1}(f)}{\pi^{2s+2}} \binom{t+1}{t-2s} \\
&\times \sum_{\ell=0}^{t-2s} \binom{t-2s}{\ell} k^{t-2s-\ell} \alpha_{w+p-k, \ell} \\
&+ \frac{o(N^{-p})}{n^{2q+r+2}}.
\end{aligned} \tag{2.14}$$

It remains to notice that $\alpha_{w+p-k, \ell} = 0$ for $0 \leq \ell < w+p-k$ (see [23]). Hence, $t-2s \geq w+p-k$, and $s \leq \lfloor (t-w)/2 \rfloor$. \square

Next theorem reveals the asymptotic behavior of the MTR-approximations for $|x| < 1$ without specifying the selection of parameters τ^c and τ^s .

Theorem 2.1 [12] *Assume $f \in C^{2q+p+2}[-1, 1]$, $f^{(2q+p+2)} \in BV[-1, 1]$, $q \geq 0$, $p \geq 1$, and*

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1. \tag{2.15}$$

Let θ_k , $k = 1, \dots, p$ be defined by (2.1). Then, the following estimates hold for $|x| < 1$ as $N \rightarrow \infty$

$$\begin{aligned}
R_{N,p}^{\cos}(f, \theta, x) &= A_{2q+1}(f) \frac{(-1)^{N+p+1}}{N^{2q+p+2} 2^{p+1} \pi^{2q+2} (2q+1)!} \\
&\times \frac{\cos \frac{\pi x}{2} (2N-p+1)}{\cos^{p+1} \frac{\pi x}{2}} h_{p,2q+1}(\tau) + o(N^{-2q-p-2}),
\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
R_{N,p}^{\sin}(f, \theta, x) &= B_{2q+1}(f) \frac{(-1)^{N+p}}{N^{2q+p+2} 2^{p+1} \pi^{2q+2} (2q+1)!} \\
&\times \frac{\sin \frac{\pi x}{2} (2N-p)}{\cos^{p+1} \frac{\pi x}{2}} h_{p,2q+1}(\tau) + o(N^{-2q-p-2}),
\end{aligned} \tag{2.17}$$

where

$$h_{p,m}(\tau) = \sum_{k=0}^p (-1)^k \gamma_k(\tau) (m+p-k)!. \tag{2.18}$$

Proof. We prove only estimate (2.16). Estimate (2.17) can be handled similarly.

Taking into account that $\theta_k \rightarrow 1$ as $N \rightarrow \infty$, we estimate only the sums on the right-hand side of (0.36). By the Abel transformation, we get

$$\begin{aligned} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta, \hat{f}^c) e^{\pm i\pi n x} &= - \frac{e^{\pm i\pi(N+1)x}}{(1 + e^{\pm i\pi x})} \Delta_N^p(\theta, \hat{f}^c) \\ &\quad - \frac{e^{\pm i\pi(N+1)x}}{(1 + e^{\pm i\pi x})^2} \Delta_N^1(\hat{\Delta}^p(\theta, \hat{f}^c)) \\ &\quad + \frac{1}{(1 + e^{\pm i\pi x})^2} \sum_{n=N+1}^{\infty} \Delta_n^2(\hat{\Delta}^p(\theta, \hat{f}^c)) e^{\pm i\pi n x}. \end{aligned} \quad (2.19)$$

Lemma 2.1 estimates sequences $\Delta_N^p(\theta, \hat{f}^c)$, $\Delta_N^1(\hat{\Delta}^p(\theta, \hat{f}^c))$ and $\Delta_n^2(\hat{\Delta}^p(\theta, \hat{f}^c))$ as $N \rightarrow \infty$ and $n \geq N + 1$. It shows that for $r = 1$ and $w = 2$, we have

$$\Delta_n^2(\hat{\Delta}^p(\theta, \hat{f}^c)) = \frac{o(N^{-p})}{n^{2q+3}}, \quad (2.20)$$

and the third term in the right-hand side of (2.19) is $o(N^{-p-2q-2})$. Then, with $r = 1$ and $w = 1$, we have

$$\Delta_N^1(\hat{\Delta}^p(\theta, \hat{f}^c)) = O(N^{-p-2q-3}), \quad (2.21)$$

and the second term is $O(N^{-p-2q-3})$. Finally, using the exact estimate for $\Delta_N^p(\theta, \hat{f}^c)$, we derive

$$\begin{aligned} \Delta_N^p(\theta, \hat{f}^c) &= A_{2q+1}(f) \frac{(-1)^N}{N^{p+2q+2} \pi^{2q+2}} \sum_{k=0}^p \gamma_k(\tau) \binom{2q+p-k+1}{2q+1} \alpha_{p-k, p-k} \\ &\quad + O(N^{-2q-p-3}), \end{aligned} \quad (2.22)$$

which completes the proof as

$$\alpha_{p-k, p-k} = (-1)^{p-k} (p-k)!. \quad (2.23)$$

□

Note that Theorem 2.1 is valid also for $p = 0$, which corresponds to the modified Fourier expansions (compare with Theorem 0.4). In that case, the exact constants of the main terms in (2.16) and (2.17) coincide with the similar estimate in [9] (Theorem 2.22, page 29).

Theorem 2.1 shows that (see (0.41))

$$R_{N,p}^q(f, \theta^c, \theta^s, x) = O(N^{-2q-p-2}) \quad (2.24)$$

if parameters θ^c and θ^s are defined by (0.45). We see improvement in convergence rate by factor $O(N^p)$ and this result is obtained without specifying parameters τ^c and τ^s .

Let us compare the modified Fourier expansions and MTR-approximations for a specific smooth function. Consider the following one

$$f(x) = (1 - x^2)^2 \sin(x - 1). \quad (2.25)$$

for which

$$\begin{aligned} f(1) &= f(-1) = 0, \quad f'(1) = f'(-1) = 0, \\ f''(1) &= 0, \quad f''(-1) = -8 \sin(2), \\ f'''(1) &= 24, \quad f'''(-1) = 24 \cos(2) + 24 \sin(2). \end{aligned} \quad (2.26)$$

Hence, the function obeys the first $q = 1$ derivative conditions (0.15).

Figures 2.1 and 2.2 show the behaviors of $|R_N(f, x)|$ ($p = 0$) and $|R_{N,p}(f, x)|$ ($p = 1, 2, 3, 4$), respectively, on interval $[-0.7, 0.7]$ for $N = 64$ while approximating (2.25). We used $\tau_k^c = \tau_k^s = k$, $k = 1, \dots, p$ in the rational approximations.

According to the results of Theorem 2.1, as bigger is the value of p as higher is the accuracy of the corresponding approximations. We observe it empirically. We see that $\max_{[-0.7, 0.7]} |R_{N,p}(f, x)|$ is $3 \cdot 10^{-8}$ for $p = 0$, is $1.6 \cdot 10^{-9}$ for $p = 1$, is $9 \cdot 10^{-11}$ for $p = 2$, is $6.6 \cdot 10^{-12}$ for $p = 3$ and is $5.7 \cdot 10^{-13}$ for $p = 4$.

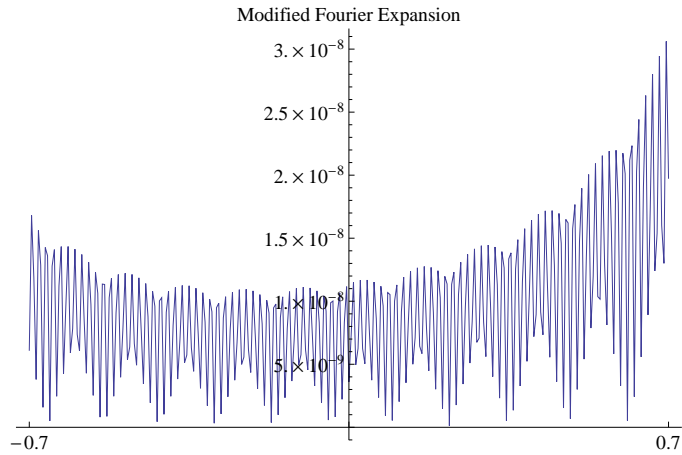


Figure 2.1: The graph of $|R_N(f, x)|$ on $[-0.7, 0.7]$ for $N = 64$ while approximating (2.25) by the modified Fourier expansion (0.3).

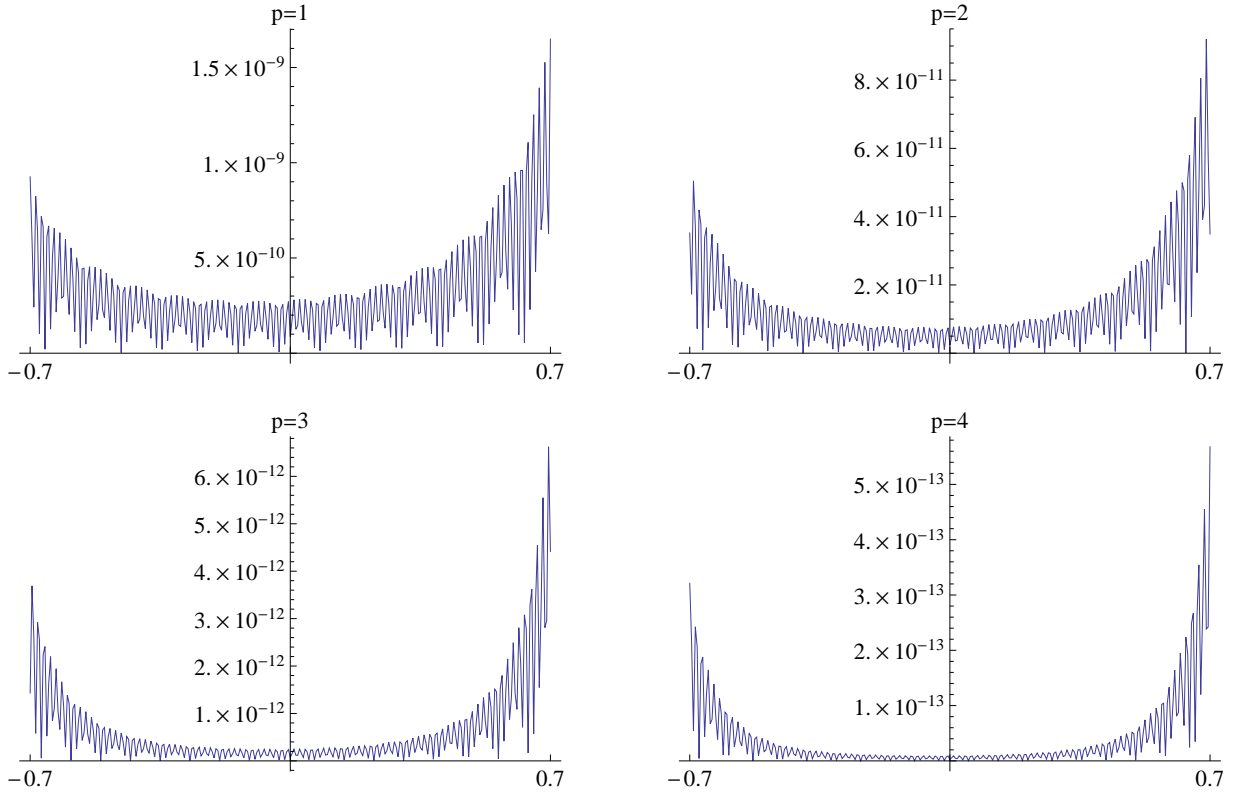


Figure 2.2: The graphs of $|R_{N,p}(f, \theta^c, \theta^s, x)|$ on $[-0.7, 0.7]$ for $N = 64$ and different p while approximating (2.25) by the MTR-approximations with $\tau_k^c = \tau_k^s = k$, $k = 1, \dots, p$.

Can we improve the accuracy of the rational approximations by appropriate selection of parameters τ^s and τ^c ? Further in this section, we give positive answer to this question and show how the optimal values can be chosen.

Estimates of Theorem 2.1 show that improvement can be achieved if parameters are chosen such that $\tau^s = \tau^c = \tau$ and

$$h_{p,2q+1}(\tau) = 0. \quad (2.27)$$

By looking into the definition of $h_{p,2q+1}(\tau)$, we observe that condition (2.27) can be achieved, for example, if

$$\gamma_k(\tau) = \binom{p}{k} \frac{(2q+1+p)!}{(2q+1+p-k)!} \mathcal{Q}_r(k), \quad (2.28)$$

where $\mathcal{Q}_r(k)$ is a polynomial of order $r \leq p-1$

$$\mathcal{Q}_r(k) = \sum_{j=0}^r c_j k^j, \quad c_0 = 1, \quad (2.29)$$

with unknown coefficients c_j , $j = 1, \dots, r$. Then, condition (2.27) follows from the well known

identity

$$\sum_{k=0}^p (-1)^k \binom{p}{k} k^j = 0, \quad j < p. \quad (2.30)$$

Further, we determine the values of c_j , $j = 1, \dots, r$ for improved convergence of the rational approximations. Next result is an immediate consequence of those observations and estimates of Theorem 2.1.

Theorem 2.2 [12] *Let $f \in C^{2q+p+2}[-1, 1]$, and $f^{(2q+p+2)} \in BV[-1, 1]$, $q \geq 0$, $p \geq 1$. Assume the following polynomial*

$$\sum_{k=0}^p \binom{p}{k} \frac{\mathcal{Q}_r(p-k)}{(2q+1+k)!} (-1)^k x^k \quad (2.31)$$

has only real-valued and non-zero roots $x = z_k$, $k = 1, \dots, p$ and let

$$\theta_k^c = \theta_k^s = 1 - \frac{z_k}{N}, \quad k = 1, \dots, p. \quad (2.32)$$

Then, the following estimate holds for $|x| < 1$

$$R_{N,p}^q(f, \theta^c, \theta^s, x) = o(N^{-2q-p-2}), \quad N \rightarrow \infty. \quad (2.33)$$

Theorem 2.2 is valid only, if for a given p and q , polynomial (2.31) has only real-valued and non-zero roots. Further, we clarify this statement by showing those cases when it is true.

By imposing extra smoothness on the underlying functions, we derive more precise estimate of (2.33). First, we need estimates for $\Delta_N^w(\hat{\Delta}^p(\theta, \hat{f}^c))$ and $\Delta_N^w(\hat{\Delta}^p(\theta, \hat{f}^s))$.

Lemma 2.2 [12] *Assume $f \in C^{2q+p+\lceil \frac{p+1}{2} \rceil+2}[-1, 1]$, $f^{(2q+p+\lceil \frac{p+1}{2} \rceil+2)} \in BV[-1, 1]$, $q \geq 0$, $p \geq 1$, and*

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1. \quad (2.34)$$

Assume polynomial (2.31) has only real-valued and non-zero roots $x = z_k$, $k = 1, \dots, p$ and θ_k is defined by (2.1) with $\tau_k = z_k$. Let $w \leq p$, when w and p have the same parity, and $w \leq p+1$, otherwise. Then, the following estimates hold as $N \rightarrow \infty$

$$\begin{aligned} \Delta_N^w(\hat{\Delta}^p(\theta, \hat{f}^c)) &= \frac{(-1)^N}{N^p} \sum_{j=0}^r c_j \sum_{t=2q+\lceil \frac{p+w-j+1}{2} \rceil}^{2q+\lceil \frac{p+1}{2} \rceil+1} \frac{1}{N^{t+2}} \sum_{s=q}^{\lceil \frac{t-w}{2} \rceil} \frac{A_{2s+1}(f)}{\pi^{2s+2}(2s+1)!} \sigma_{s,t,j}(w) \\ &+ o(N^{-2q-p-\lceil \frac{p+1}{2} \rceil-3}), \end{aligned} \quad (2.35)$$

and

$$\begin{aligned} \Delta_N^w(\hat{\Delta}^p(\theta, \hat{f}^s)) &= \frac{(-1)^{N+1}}{N^p} \sum_{j=0}^r c_j \sum_{t=2q+\lfloor \frac{p+w-j+1}{2} \rfloor}^{2q+\lfloor \frac{p+1}{2} \rfloor+1} \frac{1}{N^{t+2}} \sum_{s=q}^{\lfloor \frac{t-w}{2} \rfloor} \frac{B_{2s+1}(f)}{\pi^{2s+2}(2s+1)!} \tilde{\sigma}_{s,t,j}(w) \\ &+ o(N^{-2q-p-\lfloor \frac{p+1}{2} \rfloor-3}), \end{aligned} \quad (2.36)$$

where

$$\sigma_{s,t,j}(w) = (2q+p+1)! \sum_{k=0}^p \binom{p}{k} \beta_{k,s,t}(w) \frac{(p-k+t+1)!}{(2q+p+1-k)!} k^j, \quad (2.37)$$

and

$$\tilde{\sigma}_{s,t,j}(w) = (2q+p+1)! \sum_{k=0}^p \binom{p}{k} \tilde{\beta}_{k,s,t}(w) \frac{(p-k+t+1)!}{(2q+p+1-k)!} k^j, \quad (2.38)$$

with β and $\tilde{\beta}$ defined in Lemma 2.1.

Proof. We prove only (2.35). For that, we prove the following properties of the $\sigma_{s,t,j}(w)$

$$\sigma_{s,t,j}(w) = 0, \quad 2q \leq t \leq 2q + \left\lfloor \frac{w+p-j+1}{2} \right\rfloor - 1, \quad (2.39)$$

$$q \leq s \leq \left\lfloor \frac{t-w}{2} \right\rfloor, \quad (2.40)$$

and

$$\sigma_{s,t,j}(w) = 0, \quad t = 2q + \left\lfloor \frac{w+p-j+1}{2} \right\rfloor, \quad (2.41)$$

$$q < s \leq \left\lfloor \frac{t-w}{2} \right\rfloor. \quad (2.42)$$

Taking into account the definition of $\beta_{k,s,t}(w)$ (see (2.7)) We have

$$\begin{aligned} \sigma_{s,t,j}(w) &= (2q+p+1)! \sum_{k=0}^p \binom{p}{k} \frac{(p-k+t+1)!}{(2q+p-k+1)!} \\ &\times \sum_{\ell=w}^{t-2s} \frac{k^{t+j-2s-\ell}}{(t-2s-\ell)!(p-k+\ell)!} \sum_{u=0}^{w+p-k} (-1)^u u^{p-k+\ell} \binom{w+p-k}{u}. \end{aligned} \quad (2.43)$$

Then,

$$\begin{aligned} \sigma_{s,t,j}(w) &= (2q+p+1)! (-1)^{p+w} \sum_{k=0}^p (-1)^k \binom{p}{k} \frac{(p-k+t+1)!}{(2q+p-k+1)!} \\ &\times \sum_{u=w}^{t-2s} k^{t+j-2s-u} \frac{(w+p-k)!}{(t-2s-u)!(p-k+u)!} S(p-k+u, p-k+w), \end{aligned} \quad (2.44)$$

where $S(n, k)$ are the Stirling numbers of the second kind (see [36]). Applying the following well-known property ([36]) of the Stirling numbers

$$S(k + m, k) = \sum_{t=0}^m \binom{k + m}{m + t} c_t(m), \quad m \geq 0, \quad (2.45)$$

where $c_t(\alpha)$ are the associated Stirling numbers of the second kind, we can write

$$S(p - k + u, p - k + w) = \sum_{r=0}^{u-w} \binom{p - k + u}{u - w + r} c_r(u - w). \quad (2.46)$$

Thus, for $\sigma_{s,t,j}(w)$, we obtain

$$\begin{aligned} \sigma_{s,t,j}(w) &= (2q + p + 1)!(-1)^{p+w} \sum_{u=0}^{t-2s-w} \frac{1}{(t - 2s - u - w)!} \sum_{r=0}^u \frac{c_r(u)}{(u + r)!} \\ &\times \sum_{k=0}^p k^{t+j-2s-u-w} (-1)^k \binom{p}{k} \frac{(p - k + t + 1)!(p - k + w)!}{(2q + p - k + 1)!(p - k + w - r)!}. \end{aligned} \quad (2.47)$$

It remains to notice that the following expression

$$k^{t+j-2s-u-w} \frac{(p - k + t + 1)!(p - k + w)!}{(2q + p - k + 1)!(p - k + w - r)!} \quad (2.48)$$

is a $(2t + j - 2s - u + r - 2q - w)$ -degree polynomial of k , and hence, we can write

$$k^{t+j-2s-u-w} \frac{(p - k + t + 1)!(p - k + w)!}{(2q + p - k + 1)!(p - k + w - r)!} = \sum_{m=0}^{2t+j-2s-u+r-2q-w} d_m k^m \quad (2.49)$$

with some coefficients d_m . Therefore,

$$\begin{aligned} \sigma_{s,t,j}(w) &= (2q + p + 1)!(-1)^{p+w} \sum_{u=0}^{t-2s-w} \frac{1}{(t - 2s - u - w)!} \\ &\times \sum_{r=0}^u \frac{c_r(u)}{(u + r)!} \sum_{m=0}^{2t+j-2s-u+r-2q-w} d_m \alpha_{m,p}, \end{aligned} \quad (2.50)$$

where $\alpha_{m,p}$ are defined by (2.9). It is easy to verify that $2t + j - 2s - u + r - 2q - w < p$, which completes the proof as $\alpha_{m,p} = 0$ for $m < p$. Second, in view of (2.50), we similarly prove (2.41).

By taking $n = N$ in (2.5), and using (2.28), we get

$$\begin{aligned} \Delta_N^w(\hat{\Delta}^p(\theta, \hat{f}^c)) &= \frac{(-1)^N}{N^p} \sum_{j=0}^r c_j \sum_{t=2q+w}^{2q+\lceil \frac{p+1}{2} \rceil + 1} \frac{1}{N^{t+2}} \sum_{s=q}^{\lceil \frac{t-w}{2} \rceil} \frac{A_{2s+1}(f)}{\pi^{2s+2}(2s+1)!} \sigma_{s,t,j}(w) \\ &+ o(N^{-2q-p-\lceil \frac{p+1}{2} \rceil - 3}). \end{aligned} \quad (2.51)$$

This completes the proof in view of (2.39) and (2.41). \square

Further, in Theorems 2.3 and 2.4, we show that the pointwise convergence rate of the rational approximations depend on the asymptotic of $\Delta_N^0(\hat{\Delta}^p(\theta, \hat{f}^c))$ and $\Delta_N^0(\hat{\Delta}^p(\theta, \hat{f}^s))$.

From the other side, Lemma 2.2 reveals that the convergence rates of those sequences depend on the value of (as $w = 0$)

$$\left[\frac{p-j+1}{2} \right]. \quad (2.52)$$

When p is odd, for the highest power of $1/N$, parameter j can be only $j = 0$. It means that $\mathcal{Q}_r(k) \equiv 1$. When p is even, parameter j can be $j \leq 1$ which means that $\mathcal{Q}_r(k) = 1 + c_1 k$. Parameter c_1 we determine later.

Next theorem unveils the convergence rate of the MTR-approximations for odd values of p , when $\mathcal{Q}_r(k) = \mathcal{Q}_0(k) \equiv 1$. Note, that in this case, the roots of polynomial (2.31) coincide with the roots of generalized Laguerre polynomial $L_p^{(2q+1)}(x)$.

Theorem 2.3 [12] *Let parameter $p \geq 1$ be odd, $f \in C^{2q+p+\frac{p+1}{2}+2}[-1, 1]$, $q \geq 0$, $f^{(2q+p+\frac{p+1}{2}+2)} \in BV[-1, 1]$ and*

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1. \quad (2.53)$$

Let θ_k , $k = 1, \dots, p$ be defined by (2.1), where τ_k be the roots of the generalized Laguerre polynomial $L_p^{(2q+1)}(x)$. Then, the following estimates hold for $|x| < 1$ as $N \rightarrow \infty$

$$\begin{aligned} R_{N,p}^{\cos}(f, \theta, x) &= A_{2q+1}(f) \frac{(-1)^{N+1}}{N^{2q+p+\frac{p+1}{2}+2} \pi^{2q+2} 2^{p+1}} \\ &\times \left(\frac{\cos \frac{\pi x}{2} (2N-p+1)}{\cos^{p+1} \frac{\pi x}{2}} \sigma_{q, 2q+\frac{p+1}{2}, 0}(0) \right. \\ &\quad \left. + \frac{\cos \frac{\pi x}{2} (2N-p)}{2 \cos^{p+2} \frac{\pi x}{2}} \sigma_{q, 2q+\frac{p+1}{2}, 0}(1) \right) \\ &+ o(N^{-2q-p-\frac{p+1}{2}-2}), \end{aligned} \quad (2.54)$$

and

$$\begin{aligned} R_{N,p}^{\sin}(f, \theta, x) &= B_{2q+1}(f) \frac{(-1)^N}{N^{2q+p+\frac{p+1}{2}+2} \pi^{2q+2} 2^{p+1}} \\ &\times \left(\frac{\sin \frac{\pi x}{2} (2N-p)}{\cos^{p+1} \frac{\pi x}{2}} \tilde{\sigma}_{q, 2q+\frac{p+1}{2}, 0}(0) \right. \\ &\quad \left. + \frac{\sin \frac{\pi x}{2} (2N-p-1)}{2 \cos^{p+2} \frac{\pi x}{2}} \tilde{\sigma}_{q, 2q+\frac{p+1}{2}, 0}(1) \right) \\ &+ o(N^{-2q-p-\frac{p+1}{2}-2}), \end{aligned} \quad (2.55)$$

where σ and $\tilde{\sigma}$ are defined in Lemma 2.2.

Proof. We estimate $R_{N,p}^{cos}(f, \theta, x)$ (see (0.36)) only. The error $R_{N,p}^{sin}(f, \theta, x)$ can be estimated similarly. Taking into account that $\theta_k \rightarrow 1$ as $N \rightarrow \infty$, we estimate only the sums on the right-hand side of (0.36).

An application of the Abel transformation to the sums of $R_{N,p}^{cos}(f, \theta, x)$ leads to the following expansion of the error

$$\begin{aligned} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta, \hat{f}^c) e^{\pm i\pi n x} &= -\frac{e^{\pm i\pi(N+1)x}}{(1 + e^{\pm i\pi x})} \Delta_N^p(\theta, \hat{f}^c) - \frac{e^{\pm i\pi(N+1)x}}{(1 + e^{\pm i\pi x})^2} \Delta_N^1(\hat{\Delta}^p(\theta, \hat{f}^c)) \\ &\quad - e^{\pm i\pi(N+1)x} \sum_{w=2}^{\frac{p+1}{2}+1} \frac{\Delta_N^w(\hat{\Delta}^p(\theta, \hat{f}^c))}{(1 + e^{\pm i\pi x})^{w+1}} \\ &\quad + \frac{1}{(1 + e^{\pm i\pi x})^{\frac{p+1}{2}+2}} \sum_{n=N+1}^{\infty} \Delta_n^{\frac{p+1}{2}+2}(\hat{\Delta}^p(\theta, \hat{f}^c)) e^{\pm i\pi n x}. \end{aligned} \quad (2.56)$$

According to Lemma 2.1, we have

$$\Delta_n^{\frac{p+1}{2}+2}(\hat{\Delta}^p(\theta, \hat{f}^c)) = \frac{o(N^{-p})}{n^{2q+\frac{p+1}{2}+3}}, \quad N \rightarrow \infty, \quad n \geq N+1, \quad (2.57)$$

and hence the last term on the right-hand side of (2.56) is $o(N^{-2q-p-\frac{p+1}{2}-2})$ as $N \rightarrow \infty$.

It follows from Lemma 2.2 that the third term in (2.56) is $O(N^{-2q-p-\frac{p+1}{2}-3})$ as $N \rightarrow \infty$.

Therefore,

$$\begin{aligned} R_{N,p}^{cos}(f, \theta, x) &= -\left(\frac{e^{i\pi(N+1)x}}{2(1 + e^{i\pi x})^{p+1}} + \frac{e^{-i\pi(N+1)x}}{2(1 + e^{-i\pi x})^{p+1}} \right) \Delta_N^p(\theta, \hat{f}^c) \\ &\quad - \left(\frac{e^{i\pi(N+1)x}}{2(1 + e^{i\pi x})^{p+2}} + \frac{e^{-i\pi(N+1)x}}{2(1 + e^{-i\pi x})^{p+2}} \right) \Delta_N^1(\hat{\Delta}^p(\theta, \hat{f}^c)) \\ &\quad + o(N^{-2q-p-\frac{p}{2}-2}). \end{aligned} \quad (2.58)$$

By Lemma 2.2, we have

$$\begin{aligned} \Delta_N^p(\theta, \hat{f}^c) &= \frac{(-1)^N}{N^p} \sum_{t=2q+\frac{p+1}{2}}^{2q+\frac{p+1}{2}+1} \frac{1}{N^{t+2}} \sum_{s=q}^{\lfloor \frac{t}{2} \rfloor} \frac{A_{2s+1}(f)}{\pi^{2s+2}} \sigma_{s,t,0}(0) \\ &\quad + o(N^{-2q-p-\frac{p+1}{2}-3}) \\ &= \frac{(-1)^N}{(\pi N)^{2q+2} N^{p+\frac{p+1}{2}}} \sum_{s=q}^{q+\lfloor \frac{p+1}{4} \rfloor} \frac{A_{2s+1}(f)}{\pi^{2s}} \sigma_{s,2q+\frac{p+1}{2},0}(0) \\ &\quad + o(N^{-2q-p-\frac{p+1}{2}-2}), \end{aligned} \quad (2.59)$$

and

$$\begin{aligned}
\Delta_N^1(\hat{\Delta}^p(\theta, \hat{f}^c)) &= \frac{(-1)^N}{N^p} \sum_{t=2q+\frac{p+1}{2}}^{2q+\frac{p+1}{2}+1} \frac{1}{N^{t+2}} \sum_{s=q}^{\lfloor \frac{t-1}{2} \rfloor} \frac{A_{2s+1}(f)}{\pi^{2s+2}} \sigma_{s,t,0}(1) \\
&\quad + o(N^{-2q-p-\frac{p+1}{2}-3}) \\
&= \frac{(-1)^N}{(\pi N)^{2q+2} N^{p+\frac{p+1}{2}}} \sum_{s=q}^{q+\lfloor \frac{p-1}{4} \rfloor} \frac{A_{2s+1}(f)}{\pi^{2s}} \sigma_{s,2q+\frac{p+1}{2},0}(1) \\
&\quad + o(N^{-2q-p-\frac{p+1}{2}-2}).
\end{aligned} \tag{2.60}$$

Taking into account that (see (2.41))

$$\sigma_{s,2q+\frac{p+1}{2},0}(0) = \sigma_{s,2q+\frac{p+1}{2},0}(1) = 0, \quad s > q, \tag{2.61}$$

we conclude that $\sigma_{s,2q+\frac{p+1}{2},0}(0)$ and $\sigma_{s,2q+\frac{p+1}{2},0}(1)$ are nonzero only for $s = q$ which leads to the following estimates

$$\begin{aligned}
\Delta_N^p(\theta, \hat{f}^c) &= A_{2q+1}(f) \frac{(-1)^N}{(\pi N)^{2q+2} N^{p+\frac{p+1}{2}}} \sigma_{q,2q+\frac{p+1}{2},0}(0) \\
&\quad + o(N^{-2q-p-\frac{p+1}{2}-2}),
\end{aligned} \tag{2.62}$$

and

$$\begin{aligned}
\Delta_N^1(\hat{\Delta}^p(\theta, \hat{f}^c)) &= A_{2q+1}(f) \frac{(-1)^N}{(\pi N)^{2q+2} N^{p+\frac{p+1}{2}}} \sigma_{q,2q+\frac{p+1}{2},0}(1) \\
&\quad + o(N^{-2q-p-\frac{p+1}{2}-2}).
\end{aligned} \tag{2.63}$$

From here, we get

$$\begin{aligned}
R_{N,p}^{\cos}(f, \theta, x) &= A_{2q+1}(f) \frac{(-1)^{N+1}}{\pi^{2q+2} N^{2q+p+\frac{p}{2}+2}} \\
&\quad \times \left(\sigma_{q,2q+\frac{p+1}{2},0}(0) \operatorname{Re} \left[\frac{e^{i\pi(N+1)x}}{(1+e^{i\pi x})^{p+1}} \right] \right. \\
&\quad \quad \left. + \sigma_{q,2q+\frac{p+1}{2},0}(1) \operatorname{Re} \left[\frac{e^{i\pi(N+1)x}}{(1+e^{i\pi x})^{p+2}} \right] \right) \\
&\quad + o(N^{-2q-p-\frac{p+1}{2}-2}),
\end{aligned} \tag{2.64}$$

which completes the proof. \square

Note, that for even functions, Theorem 2.3 coincides with Theorem 4.3 of [29].

Figure 2.3 shows the result of approximation of (2.25) by the MTR-approximations with optimal parameters $\tau_k^c = \tau_k^s$, $k = 1, \dots, p$ as the roots of $L_p^{(2q+1)}(x)$. We see better accuracy on

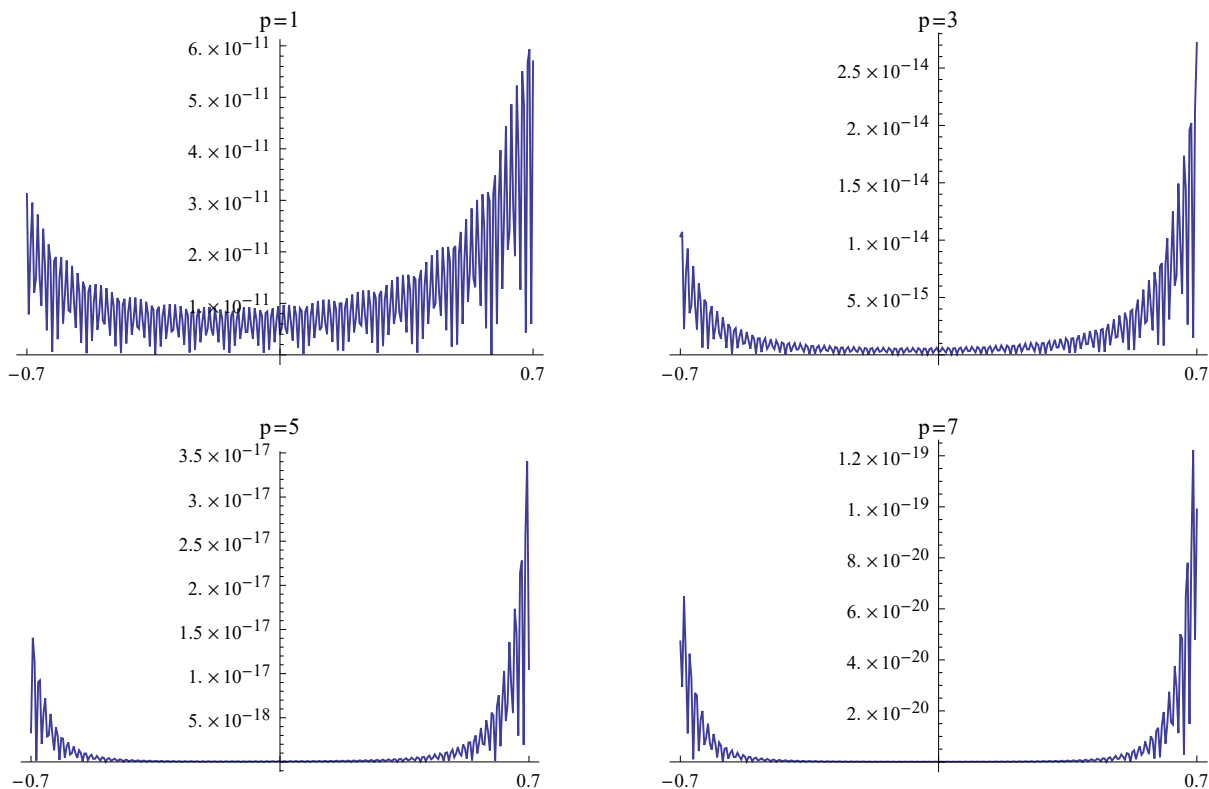


Figure 2.3: The graphs of $|R_{N,p}(f, \theta^c, \theta^s, x)|$ on interval $[-0.7, 0.7]$ for $N = 64$ while approximating (2.25) by MTR-approximations. Parameters τ_k^c and τ_k^s , $k = 1, \dots, p$ are the roots of $L_p^{(2q+1)}(x)$ which are optimal for odd p on $|x| < 1$ (see Theorem 2.3).

$[-0.7, 0.7]$ compared to non-optimal parameters as in Figure 2.2. For $p = 1$, the improvement is almost 25 times, and for $p = 3$, the improvement is almost 240 times.

Next theorem deals with even values of p . As we mentioned above, the best convergence rate is possible if $\mathcal{Q}_r(k) = \mathcal{Q}_1(k) = 1 + c_1 k$ and τ_k , $k = 1, \dots, p$ are the roots of (2.31). We need to assume that polynomial (2.31) has only real-valued and non-zero roots $x = z_k$, $k = 1, \dots, p$ for fixed p and q . In two cases, we can prove that it is true. When $c_1 = 0$, the roots of polynomial (2.31) coincide with the roots of the generalized Laguerre polynomial $L_p^{(2q+1)}(x)$. When $c_1 = -1/(2q + 1 + p)$, the roots coincide with the ones of $L_p^{(2q)}(x)$.

We saw from our experiments (which we can't prove theoretically) that polynomial (2.31) has only real-valued and non-zero roots also for other values of parameter c_1 . However, based on our experiments, we observed that the rational approximations have almost similar accuracy for different values of c_1 while approximating smooth functions on $|x| < 1$.

Theorem 2.4 [12] Let parameter $p \geq 2$ be even, $f \in C^{2q+p+\frac{p}{2}+2}[-1, 1]$, $q \geq 0$, $f^{(2q+p+\frac{p}{2}+2)} \in BV[-1, 1]$ and

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1. \quad (2.65)$$

Assume the following polynomial

$$\sum_{k=0}^p \binom{p}{k} \frac{1 + c_1(p-k)}{(2q+1+k)!} (-1)^k x^k \quad (2.66)$$

has only real-valued and non-zero roots $x = z_k$, $k = 1, \dots, p$ and let θ_k be defined by (2.1) with $\tau_k = z_k$. Then, the following estimates hold for $|x| < 1$

$$\begin{aligned} R_{N,p}^{\cos}(f, \theta, x) &= A_{2q+1}(f) \frac{(-1)^{N+1}}{N^{2q+p+\frac{p}{2}+2} \pi^{2q+2} 2^{p+1}} \\ &\times \left(\frac{\cos \frac{\pi x}{2} (2N-p+1)}{\cos^{p+1} \frac{\pi x}{2}} \left(\sigma_{q,2q+\frac{p}{2},0}(0) + \sigma_{q,2q+\frac{p}{2},1}(0) c_1 \right) \right. \\ &\quad \left. + \frac{\cos \frac{\pi x}{2} (2N-p) \sigma_{q,2q+\frac{p}{2},1}(1)}{2 \cos^{p+2} \frac{\pi x}{2}} c_1 \right) \\ &+ o(N^{-2q-p-\frac{p}{2}-2}), \end{aligned} \quad (2.67)$$

and

$$\begin{aligned} R_{N,p}^{\sin}(f, \theta, x) &= B_{2q+1}(f) \frac{(-1)^N}{N^{2q+p+\frac{p}{2}+2} \pi^{2q+2} 2^{p+1}} \\ &\times \left(\frac{\sin \frac{\pi x}{2} (2N-p)}{\cos^{p+1} \frac{\pi x}{2}} \left(\tilde{\sigma}_{q,2q+\frac{p}{2},0}(0) + \tilde{\sigma}_{q,2q+\frac{p}{2},1}(0) c_1 \right) \right. \\ &\quad \left. + \frac{\sin \frac{\pi x}{2} (2N-p-1) \tilde{\sigma}_{q,2q+\frac{p}{2},1}(1)}{2 \cos^{p+2} \frac{\pi x}{2}} c_1 \right) \\ &+ o(N^{-2q-p-\frac{p}{2}-2}), \end{aligned} \quad (2.68)$$

where σ and $\tilde{\sigma}$ are defined in Lemma 2.2.

Proof. We prove only (2.67) and need only to estimate the sums on the right-hand side of (0.36). Likewise to (2.56), we apply the Abel transformation and get the following expansion of the error

$$\begin{aligned} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta, \hat{f}^c) e^{\pm i\pi n x} &= -\frac{e^{\pm i\pi(N+1)x}}{(1 + e^{\pm i\pi x})} \Delta_N^p(\theta, \hat{f}^c) - \frac{e^{\pm i\pi(N+1)x}}{(1 + e^{\pm i\pi x})^2} \Delta_N^1(\hat{\Delta}^p(\theta, \hat{f}^c)) \\ &- e^{\pm i\pi(N+1)x} \sum_{w=2}^{\frac{p}{2}+1} \frac{\Delta_N^w(\hat{\Delta}^p(\theta, \hat{f}^c))}{(1 + e^{\pm i\pi x})^{w+1}} \\ &+ \frac{1}{(1 + e^{\pm i\pi x})^{\frac{p}{2}+2}} \sum_{n=N+1}^{\infty} \Delta_n^{\frac{p}{2}+2}(\hat{\Delta}^p(\theta, \hat{f}^c)) e^{\pm i\pi n x}. \end{aligned} \quad (2.69)$$

Taking account Lemma 2.1, we obtain

$$\Delta_n^{\frac{p}{2}+2}(\hat{\Delta}^p(\theta, \hat{f}^c)) = \frac{o(N^{-p})}{n^{2q+\frac{p}{2}+3}}, \quad N \rightarrow \infty, \quad n \geq N+1. \quad (2.70)$$

and the last term on the right-hand side of (2.69) is $o(N^{-2q-p-\frac{p}{2}-2})$ as $N \rightarrow \infty$. According to Lemma 2.2, the third term in the right-hand side of (2.69) is $O(N^{-2q-p-\frac{p}{2}-3})$ as $N \rightarrow \infty$. Therefore,

$$\begin{aligned} R_{N,p}^{cos}(f, \theta, x) &= - \left(\frac{e^{i\pi(N+1)x}}{2(1+e^{i\pi x})^{p+1}} + \frac{e^{-i\pi(N+1)x}}{2(1+e^{-i\pi x})^{p+1}} \right) \Delta_N^p(\theta, \hat{f}^c) \\ &\quad - \left(\frac{e^{i\pi(N+1)x}}{2(1+e^{i\pi x})^{p+2}} + \frac{e^{-i\pi(N+1)x}}{2(1+e^{-i\pi x})^{p+2}} \right) \Delta_N^1(\hat{\Delta}^p(\theta, \hat{f}^c)) \\ &\quad + o(N^{-2q-p-\frac{p}{2}-2}). \end{aligned} \quad (2.71)$$

According to Lemma 2.2, we get

$$\begin{aligned} \Delta_N^p(\theta, \hat{f}^c) &= \frac{(-1)^N}{N^p} \sum_{j=0}^1 c_j \sum_{t=2q+\frac{p}{2}}^{2q+\frac{p}{2}+1} \frac{1}{N^{t+2}} \sum_{s=q}^{\lfloor \frac{t}{2} \rfloor} \frac{A_{2s+1}(f)}{\pi^{2s+2}} \sigma_{s,t,j}(0) \\ &\quad + o(N^{-2q-p-\frac{p}{2}-3}) \\ &= \frac{(-1)^N}{(\pi N)^{2q+2} N^{p+\frac{p}{2}}} \sum_{j=0}^1 c_j \sum_{s=q}^{q+\lfloor \frac{p}{4} \rfloor} \frac{A_{2s+1}(f)}{\pi^{2s}} \sigma_{s,2q+\frac{p}{2},j}(0) \\ &\quad + o(N^{-2q-p-\frac{p}{2}-2}), \end{aligned} \quad (2.72)$$

and

$$\begin{aligned} \Delta_N^1(\hat{\Delta}^p(\theta, \hat{f}^c)) &= \frac{(-1)^N}{N^p} \sum_{j=0}^1 c_j \sum_{t=2q+\lfloor \frac{p+2-j}{2} \rfloor}^{2q+\frac{p}{2}+1} \frac{1}{N^{t+2}} \sum_{s=q}^{\lfloor \frac{t-1}{2} \rfloor} \frac{A_{2s+1}(f)}{\pi^{2s+2}} \sigma_{s,t,j}(1) + o(N^{-2q-p-\frac{p}{2}-3}) \\ &= \frac{(-1)^N c_1}{(\pi N)^{2q+2} N^{p+\frac{p}{2}}} \sum_{s=q}^{q+\lfloor \frac{p-2}{4} \rfloor} \frac{A_{2s+1}(f)}{\pi^{2s}} \sigma_{s,2q+\frac{p}{2},1}(1) + o(N^{-2q-p-\frac{p}{2}-2}). \end{aligned} \quad (2.73)$$

According to (2.41), we have

$$\sigma_{s,2q+\frac{p+1}{2},j}(0) = \sigma_{s,2q+\frac{p+1}{2},j}(0) = 0, \quad s > q \quad (2.74)$$

and

$$\sigma_{s,2q+\frac{p+1}{2},1}(1) = 0, \quad s > q. \quad (2.75)$$

Hence, $\sigma_{s,2q+\frac{p+1}{2},j}(0)$, $j = 0, 1$ and $\sigma_{s,2q+\frac{p+1}{2},1}(1)$ are nonzero only for $s = q$, which leads to the following estimates

$$\begin{aligned} \Delta_N^p(\theta, \hat{f}^c) &= A_{2q+1}(f) \frac{(-1)^N}{(\pi N)^{2q+2} N^{p+\frac{p}{2}}} \left(\sigma_{q,2q+\frac{p}{2},0}(0) + \sigma_{q,2q+\frac{p}{2},1}(0) c_1 \right) \\ &\quad + o(N^{-2q-p-\frac{p}{2}-2}), \end{aligned} \quad (2.76)$$

and

$$\begin{aligned} \Delta_N^1(\hat{\Delta}^p(\theta, \hat{f}^c)) &= A_{2q+1}(f) \frac{(-1)^N c_1}{(\pi N)^{2q+2} N^{p+\frac{p}{2}}} \sigma_{q,2q+\frac{p}{2},1}(1) \\ &\quad + o(N^{-2q-p-\frac{p}{2}-2}). \end{aligned} \quad (2.77)$$

Finally, from (2.71), we get

$$\begin{aligned} R_{N,p}^{cos}(f, \theta, x) &= A_{2q+1}(f) \frac{(-1)^{N+1}}{\pi^{2q+2} N^{2q+p+\frac{p}{2}+2}} \\ &\quad \times \left(\left(\sigma_{q,2q+\frac{p}{2},0}(0) + \sigma_{q,2q+\frac{p}{2},1}(0) c_1 \right) \operatorname{Re} \left[\frac{e^{i\pi(N+1)x}}{(1 + e^{i\pi x})^{p+1}} \right] \right. \\ &\quad \left. + \sigma_{q,2q+\frac{p}{2},1}(1) \operatorname{Re} \left[\frac{e^{i\pi(N+1)x}}{(1 + e^{i\pi x})^{p+2}} \right] c_1 \right) \\ &\quad + o(N^{-2q-p-\frac{p}{2}-2}), \end{aligned} \quad (2.78)$$

which completes the proof. \square

Theorems 2.3 and 2.4 conclude that by appropriate determination of parameters τ_k^c and τ_k^s , $k = 1, \dots, p$, we get extra improvement of the convergence rate of the MTR-approximations by factor $O(N^{\lceil \frac{p+1}{2} \rceil})$ compared to Theorem 2.1. Final improvement compared to the modified expansion is by factor $O(N^{p+\lceil \frac{p+1}{2} \rceil})$.

Let us return to MTR-approximations of (2.25). Figure 2 shows the results of approximation of (2.25) by the MTR-approximations with even p . Parameters $\tau_k^c = \tau_k^s$, $k = 1, \dots, p$ are selected as the roots of $L_p^{(2q+1)}(x)$. Compared with Figure 2.2, we see better accuracy on $|x| < 1$. For $p = 2$, the improvement is almost 27 times, and for $p = 4$, the improvement is almost 200 times.

Figure 2.5 shows similar results with parameters $\tau_k^c = \tau_k^s$, $k = 1, \dots, p$ as the roots of $L_p^{(2q)}(x)$. In the next section, we will prove that those parameters provide with improved accuracy also at $x = \pm 1$ for some p and q . We see that both choices of parameters provide with similar results on $|x| < 1$.

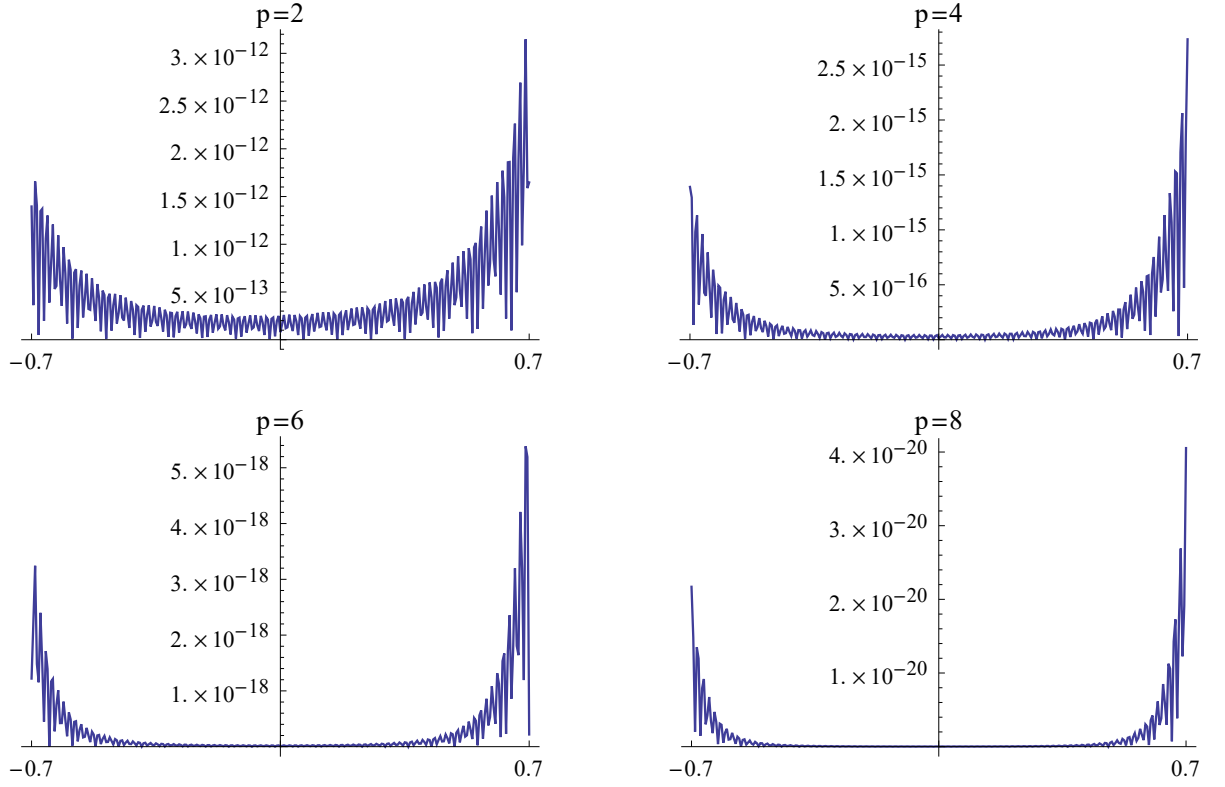


Figure 2.4: The graphs of $|R_{N,p}(f, \theta^c, \theta^s, x)|$ on interval $[-0.7, 0.7]$ for $N = 64$ while approximating (2.25). Parameters τ_k^c and τ_k^s , $k = 1, \dots, p$ are the roots of $L_p^{(2q+1)}(x)$.

Theorem 1.3 shows that the L_2 -error of the MFP-approximations depends only on the τ_k parameters which are the coefficients of the second term in the asymptotic expansion of $\theta_k = \theta_k(N)$ in terms of $\frac{1}{N}$. Paper [26] showed that the accuracy of the rational approximations by the classical Fourier system could be increased by appropriate selection of those parameters. We try the same approach for the MTR-approximations with parameters θ^c and θ^s defined by (1.61).

We omit the proof of the next theorem as it imitates the proof of 1.3.

Theorem 2.5 [11] *Let $f \in C^{(2q+p+1)}[-1, 1]$ and $f^{(2q+p+1)} \in BV[-1, 1]$, $q \geq 0$*

$p \geq 1$. Let

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1, \quad (2.79)$$

and

$$\theta_k^c = \theta_k^s = 1 - \frac{\tau_k}{N}, \quad \tau_k > 0, \quad k = 1, \dots, p. \quad (2.80)$$

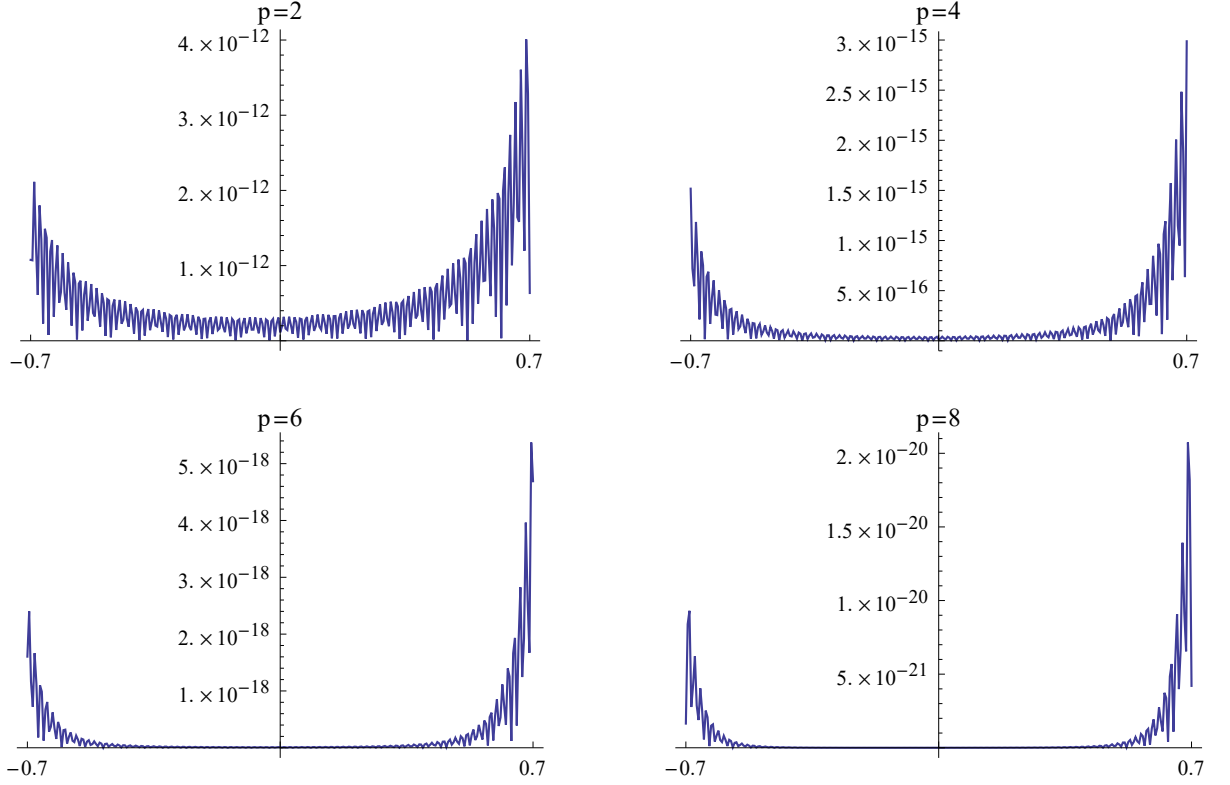


Figure 2.5: The graphs of $|R_{N,p}(f, \theta^c, \theta^s, x)|$ on interval $[-0.7, 0.7]$ for $N = 64$ while approximating (2.25). Parameters τ_k^c and τ_k^s , $k = 1, \dots, p$ are the roots of $L_p^{(2q)}(x)$.

Then,

$$\lim_{N \rightarrow \infty} N^{2q + \frac{3}{2}} \|R_{N,p}\|_{L_2} = \frac{c_{p,q}^*}{\pi^{2q+2} \sqrt{4q+3}} \sqrt{A_{2q+1}^2(f) + B_{2q+1}^2(f)}, \quad (2.81)$$

where $c_{p,q}^*$ is defined by (1.82) for all $\tau_k > 0$, $k = 1, \dots, p$.

In estimate (2.81), the constant $c_{p,q}^*$ will coincide with $c_{p,q}$ if the parameters τ_k are the roots of $L_p^{(2q+1)}(x)$. Our goal is the minimization of $c_{p,q}^*$ by an appropriate selection of parameters τ_k , $k = 1, \dots, p$ (see also [26] for a similar problem). The corresponding MTR-approximations, we call as L_2 -minimal MTR-approximations. Table 2.1 shows some of the optimal values of τ_k (see [26]) with the corresponding value of $1/c_{p,q}^*$ which shows the efficiency of the L_2 -minimal MTR-approximation in comparison with the classical expansions by the modified Fourier system with the exact asymptotic constant $c_{0,q} = 1$.

Comparison of Tables 1.3 and 2.1 shows that the L_2 -error of the L_2 -minimal MTR-approximation is smaller than the L_2 -error of the corresponding MFP-approximation. For example, in case of $p = 2$ and $q = 6$, the asymptotic improvement is almost 4.9 times.

q	0	1	2	3	4	5	6
$1/c_{1,q}^*$	5.7	12.2	18.7	25.3	31.8	38.3	44.9
τ_1	1.35	3.3	5.3	7.3	9.3	11.3	13.3
$1/c_{2,q}^*$	21.1	82.4	183.8	326	508	731	994
τ_1	2.8	5.4	3.3	4.9	6.5	14.9	17.1
τ_2	0.53	1.8	7.8	10.2	12.6	8.2	9.9
$1/c_{3,q}^*$	61.3	412	1297	2965	5667	9650	15167
τ_1	4.2	7.3	5.2	7.2	4.95	17.9	20.4
τ_2	0.3	3.2	10.1	12.7	9.1	6.4	7.9
τ_3	1.29	1.12	2.3	3.6	15.3	11.1	13.1
$1/c_{4,q}^*$	157	1704	7378	21442	49700	99492	179701
τ_1	2.2	4.6	6.9	5.4	11.4	5.2	10.6
τ_2	5.7	9.1	12.2	15.1	17.9	8.8	6.5
τ_3	0.1	2.1	3.7	9.2	7.1	20.6	23.3
τ_4	0.67	0.74	1.6	2.7	3.9	13.7	15.9

Table 2.1: Numerical values of $1/c_{p,q}^*$ for $p = 1, 2, 3, 4$ with the optimal values of τ_k , $k = 1, \dots, p$ that minimize $c_{p,q}^*$.

3. Convergence of the Optimal MTR-approximations at

$$x = \pm 1$$

In this section, we explore the pointwise convergence of the MTR-approximations at the endpoints $x = \pm 1$. Next theorem explores the convergence of the rational approximations without determining parameters τ^c and τ^s .

Theorem 3.1 [12] Assume $f \in C^{2q+p+1}[-1, 1]$, $f^{(2q+p+1)} \in BV[-1, 1]$, $q \geq 0$, $p \geq 1$, and

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1. \quad (3.1)$$

Let θ_k , $k = 1, \dots, p$ be defined by (2.1). Then, the following estimates hold

$$R_{N,p}^{cos}(f, \theta, \pm 1) = \frac{A_{2q+1}(f)}{N^{2q+1}} \frac{(-1)^p}{(2q+1)! \pi^{2q+2} \gamma_p(\tau)} h_{p,2q}(\tau) + o(N^{-2q-1}), \quad N \rightarrow \infty, \quad (3.2)$$

and

$$R_{N,p}^{sin}(f, \theta, \pm 1) = \pm \frac{B_{2q+1}(f)}{N^{2q+1}} \frac{(-1)^p}{(2q+1)! \pi^{2q+2} \gamma_p(\tau)} h_{p,2q}(\tau) + o(N^{-2q-1}), \quad N \rightarrow \infty, \quad (3.3)$$

where $h_{p,m}(\tau)$ is defined by (2.18).

Proof. In view of (0.36) and (0.37), we write

$$R_{N,p}^{cos}(f, \theta, \pm 1) = \frac{N^p}{\gamma_p(\tau)} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta, \hat{f}^c) (-1)^n, \quad (3.4)$$

and

$$R_{N,p}^{sin}(f, \theta, \pm 1) = \mp \frac{N^p}{\gamma_p(\tau)} \sum_{n=N+1}^{\infty} \Delta_n^p(\theta, \hat{f}^s) (-1)^n. \quad (3.5)$$

Now, the prove immediately follows from the estimates of Lemma 2.1 by taking $w = 0$, $r = 0$ and by recalling that $\alpha_{p-k,p-k} = (-1)^{p-k} (p-k)!$. \square

Note that this theorem is valid also for $p = 0$, which corresponds to the expansions by the modified Fourier system (compare with Theorem 0.5). Exact constants of the main terms in (3.2) and (3.3), for $p = 0$, can be found also in [6] (Theorem 3.2). We see that, in general, rational corrections don't increase the convergence rates of modified Fourier expansions at the endpoints $x = \pm 1$ without specifying appropriately parameters τ^c and τ^s . Both approaches have the same convergence rates $O(N^{-2q-1})$. Moreover, as Figure 3.1 shows, without reasonable selection of the parameters, modified Fourier expansions have better accuracy compared to "non-optimal" rational approximations at $x = \pm 1$.

Is it possible to improve the accuracy by appropriate selection of parameters τ^c and τ^s ? The answer is positive and the solution is in the estimates of Theorem 3.1. We put

$$\gamma_k(\tau) = \binom{p}{k} \frac{(2q+p)!}{(2q+p-k)!} \mathcal{P}_r(k), \quad (3.6)$$

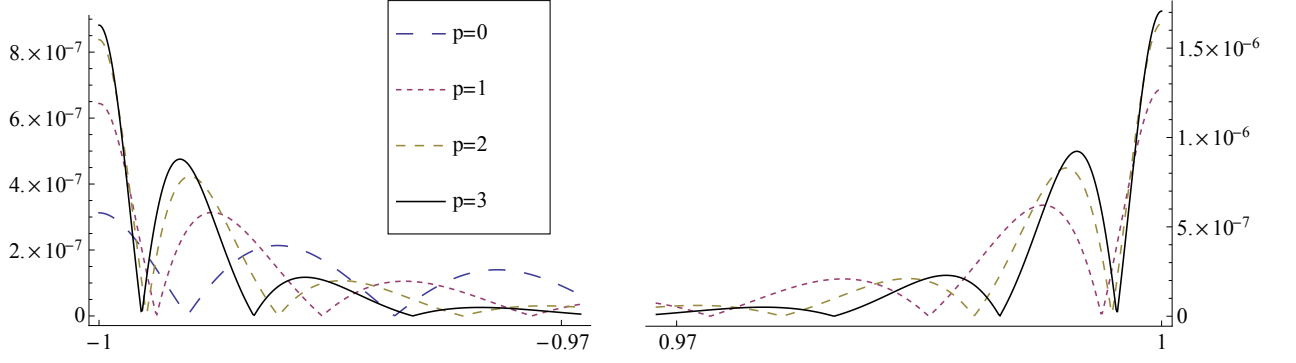


Figure 3.1: The graphs of $|R_{N,p}(f, \theta^c, \theta^s, x)|$ for $p = 0, 1, 2, 3$ and $N = 64$ while approximating (2.25) at the points $x = \pm 1$. In rational approximations, we took $\tau_k^c = \tau_k^s = k$, $k = 1, \dots, p$.

where

$$\mathcal{P}_r(k) = \sum_{j=0}^r d_j k^j, \quad d_0 = 1. \quad (3.7)$$

Now, the property $h_{p,2q}(\tau) = 0$ follows from the identity

$$\sum_{k=0}^p (-1)^k \binom{p}{k} k^j = 0, \quad j < p. \quad (3.8)$$

Next theorem is the result of these observations and Theorem 3.1.

Theorem 3.2 [12] *Let $f \in C^{2q+p+1}[-1, 1]$, $q \geq 0$, $p \geq 1$, and $f^{(2q+p+1)} \in BV[-1, 1]$. Assume the following polynomial*

$$\sum_{k=0}^p \binom{p}{k} \frac{\mathcal{P}_r(p-k)}{(2q+k)!} (-1)^k x^k \quad (3.9)$$

has only real-valued and non-zero roots $x = z_k$, $k = 1, \dots, p$ and let

$$\theta_k^c = \theta_k^s = 1 - \frac{z_k}{N}, \quad k = 1, \dots, p. \quad (3.10)$$

Then,

$$R_{N,p}^q(f, \theta^c, \theta^s, \pm 1) = o(N^{-2q-1}), \quad N \rightarrow \infty. \quad (3.11)$$

Our next goal is derivation of the exact convergence rate of (3.11).

Lemma 3.1 [12] *Assume $f \in C^{2q+p+\lceil \frac{p+1}{2} \rceil+1}[-1, 1]$, $q \geq 0$, $p \geq 1$, $f^{(2q+p+\lceil \frac{p+1}{2} \rceil+1)} \in BV[-1, 1]$*

and

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1. \quad (3.12)$$

Assume the following polynomial

$$\sum_{k=0}^p \binom{p}{k} \frac{\mathcal{P}_r(p-k)}{(2q+k)!} (-1)^k x^k \quad (3.13)$$

has only real-valued and non-zero roots $x = z_k$, $k = 1, \dots, p$ and let θ_k be defined by (2.1) with

$\tau_k = z_k$. Then, the following asymptotic expansions hold as $N \rightarrow \infty$

$$\begin{aligned} R_{N,p}^{\cos}(f, \theta, \pm 1) &= \frac{1}{\gamma_p(\tau)} \sum_{j=0}^r d_j \sum_{t=2q+\lceil \frac{p+1}{2} \rceil}^{2q+\lceil \frac{p+1}{2} \rceil} \frac{1}{N^{t+1}} \\ &\times \left(\sum_{\ell=0}^{t-2q} b_\ell \frac{(-1)^\ell}{\ell!} \sum_{s=q}^{\lfloor \frac{t-\ell}{2} \rfloor} \frac{A_{2s+1}(f)}{\pi^{2s+2}(2s+1)!} \delta_{q,t-\ell,j}(\ell) \right. \\ &\quad \left. - \sum_{s=q}^{\lfloor \frac{t}{2} \rfloor} \frac{A_{2s+1}(f)}{\pi^{2s+2}(2s+1)!} \delta_{s,t,j}(1) \right) + o(N^{-2q-\lceil \frac{p+1}{2} \rceil-1}), \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} R_{N,p}^{\sin}(f, \theta, \pm 1) &= \pm \frac{1}{\gamma_p(\tau)} \sum_{j=0}^r d_j \sum_{t=2q+\lceil \frac{p+1}{2} \rceil}^{2q+\lceil \frac{p+1}{2} \rceil} \frac{1}{N^{t+1}} \\ &\times \left(\sum_{\ell=0}^{t-2q} b_\ell \frac{(-1)^\ell}{\ell!} \sum_{s=q}^{\lfloor \frac{t-\ell}{2} \rfloor} \frac{B_{2s+1}(f)}{\pi^{2s+2}(2s+1)!} \tilde{\delta}_{q,t-\ell,j}(\ell) \right. \\ &\quad \left. - \sum_{s=q}^{\lfloor \frac{t}{2} \rfloor} \frac{B_{2s+1}(f)}{\pi^{2s+2}(2s+1)!} \tilde{\delta}_{s,t,j}(1) \right) + o(N^{-2q-\lceil \frac{p+1}{2} \rceil-1}), \end{aligned} \quad (3.15)$$

where b_ℓ is the ℓ -th Bernoulli number and

$$\delta_{s,t,j}(w) = (2q+p)! \sum_{k=0}^p \binom{p}{k} \beta_{k,s,t}(0) \frac{(p-k+t+w)!}{(2q+p-k)!} k^j, \quad (3.16)$$

and

$$\tilde{\delta}_{s,t,j}(w) = (2q+p)! \sum_{k=0}^p \binom{p}{k} \tilde{\beta}_{k,s,t}(0) \frac{(p-k+t+w)!}{(2q+p-k)!} k^j, \quad (3.17)$$

with β and $\tilde{\beta}$ defined in Lemma 2.1.

Proof. We prove only (3.14). First we prove that

$$\delta_{s,t,j}(w) = 0, \quad (3.18)$$

when

$$2q \leq t \leq 2q + \left\lfloor \frac{p+1-w-j}{2} \right\rfloor - 1, \text{ and } q \leq s \leq \left\lfloor \frac{t}{2} \right\rfloor. \quad (3.19)$$

Taking into account the definition of $\beta_{k,s,t}(w)$ (see (2.7)) and using (2.45) property of the Stirling numbers of the second kind we receive

$$\begin{aligned} \delta_{s,t,j}(w) &= (2q+p)!(-1)^p \sum_{u=0}^{t-2s} \frac{1}{(t-2s-u)!} \sum_{r=0}^u \frac{c_r(u)}{(u+r)!} \\ &\times \sum_{k=0}^p k^{t+j-2s-u} (-1)^k \binom{p}{k} \frac{(p-k+t+w)!(p-k)!}{(2q+p-k)!(p-k-r)!}. \end{aligned} \quad (3.20)$$

Similar to (2.50), we derive that

$$\delta_{s,t,j}(w) = (2q+p)!(-1)^p \sum_{u=0}^{t-2s} \frac{1}{(t-2s-u)!} \sum_{r=0}^u \frac{c_r(u)}{(u+r)!} \sum_{m=0}^{2t+j+w-2s-u+r-2q} d_m \alpha_{m,p}, \quad (3.21)$$

where $\alpha_{m,p}$ are defined by (2.9), which completes the proof as $\alpha_{m,p} = 0$ for $m < p$.

Second, in view of (3.21), we similarly prove that

$$\delta_{s,t,j}(w) = 0, \quad t = 2q + \left\lfloor \frac{p+1-j-w}{2} \right\rfloor, \quad q < s \leq \left\lfloor \frac{t}{2} \right\rfloor. \quad (3.22)$$

In view of Lemma 2.1 (with $w = 0$) and equation (3.4), we write

$$\begin{aligned} R_{N,p}^{\cos}(f, \theta, \pm 1) &= \frac{N^p}{\gamma_p(\tau)} \sum_{k=0}^p \frac{\gamma_k(\tau)}{N^k} \sum_{t=2q}^{2q+\lfloor \frac{p+1}{2} \rfloor} (p-k+t+1)! \\ &\times \sum_{s=q}^{\lfloor \frac{t}{2} \rfloor} \frac{A_{2s+1}(f)}{\pi^{2s+2}(2s+1)!} \beta_{k,s,t}(0) \sum_{n=N+1}^{\infty} \frac{1}{n^{p-k+t+2}} \\ &+ o(N^{-2q-\lfloor \frac{p+1}{2} \rfloor - 1}). \end{aligned} \quad (3.23)$$

We estimate the infinite sum on the right-hand side of (3.23) by the Euler-Maclaurin formula (see [37]). We have

$$\begin{aligned} \sum_{n=N}^{\infty} \frac{1}{n^{p-k+t+2}} &= \frac{1}{p-k+t+1} \sum_{w=0}^{2q+\lfloor \frac{p+1}{2} \rfloor - t} \binom{p-k+t+w}{w} \frac{b_w(-1)^w}{N^{p-k+t+w+1}} \\ &+ O(N^{-2q-p+k-\lfloor \frac{p+1}{2} \rfloor - 2}), \end{aligned} \quad (3.24)$$

where b_w is the w -th Bernoulli number. Then,

$$\begin{aligned} R_{N,p}^{\cos}(f, \theta, \pm 1) &= \frac{1}{\gamma_p(\tau)} \sum_{j=0}^r d_j \sum_{t=2q}^{2q+\lfloor \frac{p+1}{2} \rfloor} \frac{1}{N^{t+1}} \sum_{\ell=0}^{t-2q} b_\ell \frac{(-1)^\ell}{\ell!} \sum_{s=q}^{\lfloor \frac{t-\ell}{2} \rfloor} \frac{A_{2s+1}(f)}{\pi^{2s+2}(2s+1)!} \delta_{s,t-\ell,j}(\ell) \\ &- \frac{1}{\gamma_p(\tau)} \sum_{j=0}^r d_j \sum_{t=2q}^{2q+\lfloor \frac{p+1}{2} \rfloor - 1} \frac{1}{N^{t+2}} \sum_{r=q}^{\lfloor \frac{t}{2} \rfloor} \frac{A_{2s+1}(f)}{\pi^{2s+2}(2s+1)!} \delta_{s,t,j}(1) \\ &+ o(N^{-2q-\lfloor \frac{p+1}{2} \rfloor - 1}), \end{aligned} \quad (3.25)$$

which completes the proof in view of (3.18) and (3.19) \square

By repeating the observations of previous section, it is possible to deduce that for getting the maximal convergence rate for odd values of p , the polynomial $\mathcal{P}_r(k)$ can be at most degree-0 polynomial, $\mathcal{P}_r(k) = \mathcal{P}_0(k) \equiv 1$. For even values of p , $\mathcal{P}_r(k) = \mathcal{P}_1(k) = 1 + d_1 k$. In the first case, parameters τ_k are the roots of $L_p^{(2q)}(x)$. In the second case, if $d_1 = 0$, we get the roots of $L_p^{(2q)}(x)$ and, if $d_1 = -1/(2q + p)$, we get the roots of $L_p^{(2q-1)}(x)$. The next two theorems immediately follow from Lemma 3.1 and identity (3.22) and, we omit the proofs.

Theorem 3.3 [12] *Let parameter $p \geq 1$ be odd, $f \in C^{2q+p+\frac{p+1}{2}+2}[-1, 1]$, $q \geq 0$, $f^{(2q+p+\frac{p+1}{2}+2)} \in BV[-1, 1]$ and*

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1. \quad (3.26)$$

Let θ_k , $k = 1, \dots, p$ be defined by (2.1), where τ_k , $k = 1, \dots, p$ be the roots of the generalized Laguerre polynomial $L_p^{(2q)}(x)$. Then, the following estimates hold as $N \rightarrow \infty$

$$\begin{aligned} R_{N,p}^{\cos}(f, \theta, \pm 1) &= \frac{1}{N^{2q+\frac{p+1}{2}+1}} \frac{A_{2q+1}(f)}{\gamma_p(\tau) \pi^{2q+2} (2q+1)!} \\ &\times \left(\delta_{q, 2q+\frac{p+1}{2}, 0}(0) - \frac{1}{2} \delta_{q, 2q+\frac{p-1}{2}, 0}(1) \right) \\ &+ o(N^{-2q-\frac{p+1}{2}-1}), \end{aligned} \quad (3.27)$$

and

$$\begin{aligned} R_{N,p}^{\sin}(f, \theta, \pm 1) &= \pm \frac{1}{N^{2q+\frac{p+1}{2}+1}} \frac{B_{2q+1}(f)}{\gamma_p(\tau) \pi^{2q+2} (2q+1)!} \\ &\times \left(\tilde{\delta}_{q, 2q+\frac{p+1}{2}, 0}(0) - \frac{1}{2} \tilde{\delta}_{q, 2q+\frac{p-1}{2}, 0}(1) \right) \\ &+ o(N^{-2q-\frac{p+1}{2}-1}), \end{aligned} \quad (3.28)$$

where δ and $\tilde{\delta}$ are defined in Lemma 3.1.

Figure 3.2 shows the result of approximation of (2.25) by the rational approximations with optimal values of parameters τ_k , $k = 1, \dots, p$ as in Theorem 3.3. We see that by increasing p (p is odd), we increase the accuracy of approximations at the points $x = \pm 1$. Note that $p = 0$ corresponds to the classical expansion by the modified system and we see that in contrary to Figure 3.1, the optimal choice of parameters do have big positive impact on the accuracy.

Comparison of Theorems 2.3 and 3.3 reveals the problem of the optimal rational approximations which is in the difference of optimal values of parameters τ_k for $|x| < 1$ and $x = \pm 1$. On $|x| < 1$ and $x = \pm 1$, the optimal values are the roots of $L_p^{(2q+1)}(x)$ and $L_p^{(2q)}(x)$, respectively. The choice of $L_p^{(2q)}(x)$ will result in better accuracy on overall $[-1, 1]$ by the uniform norm, but on $|x| < 1$ the rate of convergence will be worse by factor $O(N)$.

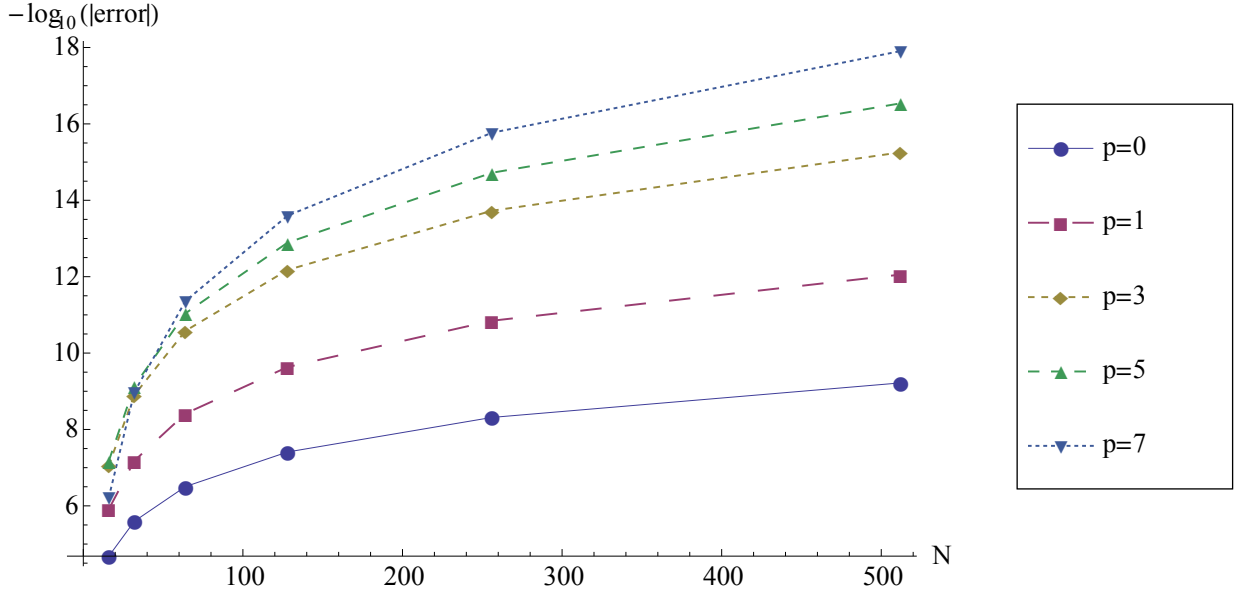


Figure 3.2: The values of $-\log_{10} \left(\max_{x=\pm 1} |R_{N,p}(f, \theta^c, \theta^s, x)| \right)$ while approximating (2.25) for different N and p . Parameters τ_k^c and τ_k^s , $k = 1, \dots, p$ are the roots of the generalized Laguerre polynomial $L_p^{(2q)}(x)$.

Next theorem explores even values of p .

Theorem 3.4 [12] *Let parameter $p \geq 2$ be even, $f \in C^{2q+p+\frac{p}{2}+2}[-1, 1]$, $q \geq 0$, $f^{(2q+p+\frac{p}{2}+2)} \in BV[-1, 1]$ and*

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1. \quad (3.29)$$

Assume the following polynomial

$$\sum_{k=0}^p \binom{p}{k} \frac{1 + d_1(p-k)}{(2q+k)!} (-1)^k x^k \quad (3.30)$$

has only real-valued and non-zero roots $x = z_k$, $k = 1, \dots, p$ and let θ_k be defined by (2.1) with

$\tau_k = z_k$. Then, the following asymptotic expansions hold

$$R_{N,p}^{\cos}(f, \theta, \pm 1) = \frac{1}{N^{2q+\frac{p}{2}+1}} \frac{A_{2q+1}(f)}{\gamma_p(\tau)\pi^{2q+2}(2q+1)!} \Phi_{q,p}(d_1) + o(N^{-2q-\frac{p}{2}-1}), N \rightarrow \infty, \quad (3.31)$$

and

$$R_{N,p}^{\sin}(f, \theta, \pm 1) = \pm \frac{1}{N^{2q+\lceil\frac{p+1}{2}\rceil+1}} \frac{B_{2q+1}(f)}{\gamma_p(\tau)\pi^{2q+2}(2q+1)!} \tilde{\Phi}_{q,p}(d_1) + o(N^{-2q-\frac{p}{2}-1}), N \rightarrow \infty, \quad (3.32)$$

where

$$\Phi_{q,p}(d_1) = \delta_{q,2q+\frac{p}{2},0}(0) - \delta_{q,2q+\frac{p}{2}-1,0}(1) + d_1 \left(\delta_{q,2q+\frac{p}{2},1}(0) - \frac{1}{2} \delta_{q,2q+\frac{p}{2}-1,1}(1) \right), \quad (3.33)$$

and

$$\tilde{\Phi}_{q,p}(d_1) = \tilde{\delta}_{q,2q+\frac{p}{2},0}(0) - \tilde{\delta}_{q,2q+\frac{p}{2}-1,0}(1) + d_1 \left(\tilde{\delta}_{q,2q+\frac{p}{2},1}(0) - \frac{1}{2} \tilde{\delta}_{q,2q+\frac{p}{2}-1,1}(1) \right), \quad (3.34)$$

with δ and $\tilde{\delta}$ defined in Lemma 3.1.

Estimates (3.31) and (3.32) are valid if polynomial (3.30) has only real-valued and nonzero roots. As we mentioned above, in two particular cases when $d_1 = 0$ and $d_1 = -1/(2q+p)$, the roots of the polynomial coincide with the roots of $L_p^{(2q)}(x)$ and $L_p^{(2q-1)}(x)$, respectively. Both Laguerre polynomials have only real-valued and positive roots and Theorem 3.4 is valid in both cases. The choice of polynomial $L_p^{(2q)}(x)$ is reasonable as it will provide with optimal rational approximation both on $|x| < 1$ (see Theorem 2.4) and at $x = \pm 1$ for some p and q .

Figure 3.3 shows the result of application of the rational approximations to function (2.25) with optimal values of parameters τ_k , $k = 1, \dots, p$ as the roots of $L_p^{(2q)}(x)$.

Now, we show that for some p and q , estimates (3.31) and (3.32) can be improved with appropriate selection of parameter d_1 . Assume that for a given p and q , it is possible to vanish $\Phi_{q,p}(d_1)$ by appropriate selection of d_1 in (3.31). Then, we derive

$$R_{N,p}^{\cos}(f, \theta^c, \theta^s, \pm 1) = o(N^{-2q-\frac{p}{2}-1}), \quad (3.35)$$

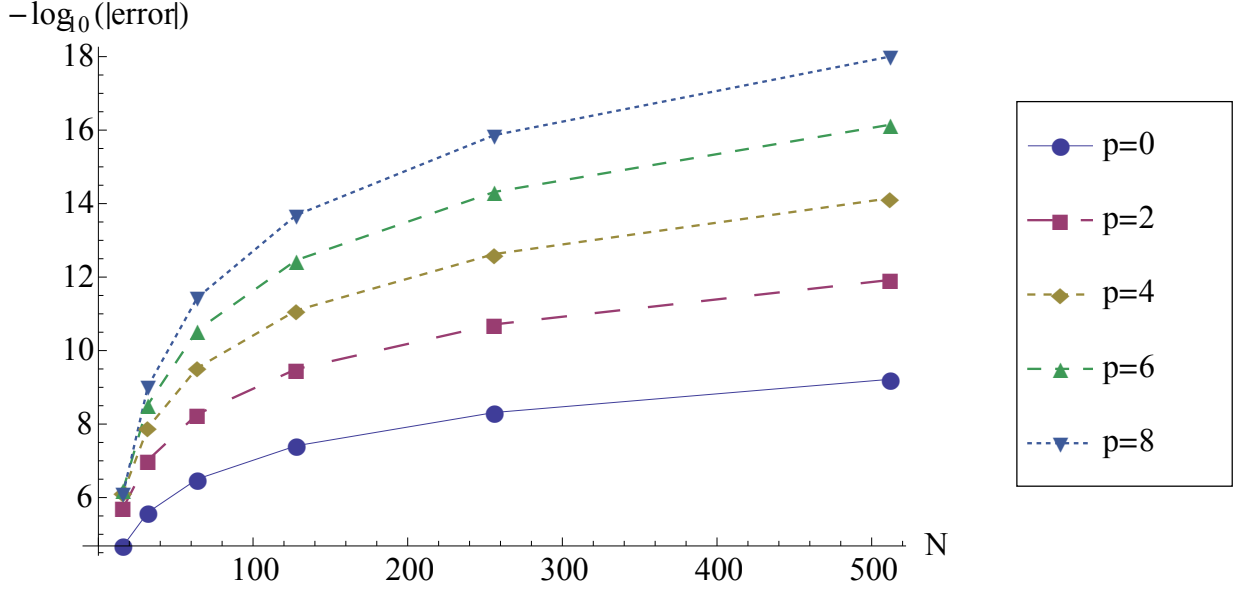


Figure 3.3: The values of $-\log_{10} \left(\max_{x=\pm 1} |R_{N,p}(f, \theta^c, \theta^s, x)| \right)$ while approximating (2.25) for different N and p . Parameters τ_k^c and τ_k^s , $k = 1, \dots, p$ are the roots of the generalized Laguerre polynomial $L_p^{(2q)}(x)$.

or

$$R_{N,p}^{cos}(f, \theta^c, \theta^s, \pm 1) = O(N^{-2q - \frac{p}{2} - 2}) \quad (3.36)$$

in case of smoother functions. Similarly, if $\tilde{\Phi}_{q,p}(d_1) = 0$ by appropriate selection of parameter d_1 , then,

$$R_{N,p}^{sin}(f, \theta^c, \theta^s, \pm 1) = o(N^{-2q - \frac{p}{2} - 1}), \quad (3.37)$$

or

$$R_{N,p}^{sin}(f, \theta^c, \theta^s, \pm 1) = O(N^{-2q - \frac{p}{2} - 2}) \quad (3.38)$$

in case of smoother functions.

The problem is that, we can not vanish both $\Phi_{q,p}(d_1)$ and $\tilde{\Phi}_{q,p}(d_1)$ simultaneously by the same d_1 . Hence, we decompose a function into even and odd parts, and perform separate optimizations in terms of parameter d_1 . In order to choose parameter d_1 appropriately, we need to have

$$\delta_{q, 2q + \frac{p}{2}, 1}(0) - \frac{1}{2} \delta_{q, 2q + \frac{p}{2} - 1, 1}(1) \neq 0, \quad (3.39)$$

and

$$\tilde{\delta}_{q,2q+\frac{p}{2},1}(0) - \frac{1}{2}\tilde{\delta}_{q,2q+\frac{p}{2}-1,1}(1) \neq 0 \quad (3.40)$$

for even and odd functions, respectively.

Then, we put

$$d_1 = d_p^{even}(q) = \frac{-\delta_{q,2q+\frac{p}{2},0}(0) + \delta_{q,2q+\frac{p}{2}-1,0}(1)}{\delta_{q,2q+\frac{p}{2},1}(0) - \frac{1}{2}\delta_{q,2q+\frac{p}{2}-1,1}(1)}, \quad (3.41)$$

and

$$d_1 = d_p^{odd}(q) = \frac{-\tilde{\delta}_{q,2q+\frac{p}{2},0}(0) + \tilde{\delta}_{q,2q+\frac{p}{2}-1,0}(1)}{\tilde{\delta}_{q,2q+\frac{p}{2},1}(0) - \frac{1}{2}\tilde{\delta}_{q,2q+\frac{p}{2}-1,1}(1)}. \quad (3.42)$$

Finally, we put $d_p^{even}(q)$ and $d_p^{odd}(q)$ into (3.30) and if that polynomials have only real-valued and nonzero roots, the optimization process will succeed. Tables 3.1 and 3.2 show that except some special cases, we can optimize estimates of Theorem 3.4.

q	0	1	2	3	4	5	6
$d_2^{even}(q)$	-1	1	1/3	1/5	1/7	1/9	1/11
τ_1	-1.41	2.71	4.26	5.89	7.57	9.28	11.01
τ_2	1.41	13.29	11.74	13.31	15.29	17.39	19.53
$d_4^{even}(q)$	-3/14	-3/2	3/10	3/22	3/34	3/46	3/58
τ_1	0.12	-27.44	2.61	3.81	5.10	6.45	7.85
τ_2	1.09	1.57	5.86	7.50	9.23	10.99	12.78
τ_3	3.46	4.44	10.85	12.72	14.76	16.86	18.98
τ_4	7.91	9.43	22.28	21.43	23.15	25.36	27.71
$d_6^{even}(q)$	-1/11	-1/5	1	1/7	1/13	1/19	1/25
τ_1	0.18	-0.74	1.96	2.90	3.95	5.07	6.25
τ_2	1.00	1.34	4.28	5.57	6.98	8.46	9.98
τ_3	2.60	3.41	7.51	9.04	10.76	12.54	14.35
τ_4	5.15	6.45	11.99	13.62	15.55	17.59	19.67
τ_5	8.96	10.74	18.54	19.91	21.87	24.08	26.38
τ_6	14.84	17.18	75.72	31.25	31.35	33.32	35.69

Table 3.1: The values of $d_p^{even}(q)$ and the roots of (3.30) for $p = 2, 4, 6$ and $0 \leq q \leq 6$.

q	0	1	2	3	4	5	6
$d_2^{odd}(q)$	—	—	1/2	1/4	1/6	1/8	1/10
τ_1	—	—	4.417	6	7.653	9.347	11.07
τ_2	—	—	13.58	14	15.68	17.65	19.73
$d_4^{odd}(q)$	$-3/20$	$-3/8$	$3/4$	$3/16$	$3/28$	$3/40$	$3/52$
τ_1	0.23	-1.95	2.70	3.87	5.14	6.48	7.87
τ_2	1.35	1.83	6.09	7.63	9.31	11.05	12.83
τ_3	3.77	4.96	11.41	13.01	14.93	16.98	19.07
τ_4	8.249	10.16	35.80	22.99	23.76	25.69	27.92
$d_6^{odd}(q)$	$-1/14$	$-1/8$	$-1/2$	$1/4$	$1/10$	$1/16$	$1/22$
τ_1	0.20	0.53	-16.59	2.94	3.98	5.09	6.27
τ_2	1.06	1.80	2.07	5.67	7.04	8.50	10.01
τ_3	2.71	3.88	4.52	9.23	10.86	12.60	14.40
τ_4	5.29	6.92	7.93	13.95	15.73	17.70	19.74
τ_5	9.14	11.23	12.62	20.58	22.21	24.28	26.52
τ_6	15.04	17.65	19.45	37.63	32.58	33.82	35.98

Table 3.2: The values of $d_p^{odd}(q)$ and the roots of (3.30) for $p = 2, 4, 6$ and $0 \leq q \leq 4$.

Figures 3.4 and 3.5 show the errors at $x = \pm 1$ while approximating (2.25) with rational approximations, where parameters τ_k^c and τ_k^s , $k = 1, \dots, p$ are selected according to Tables 3.1 and 3.2 for even and odd parts of the function, respectively. We called this approach as "optimal" in the figures. For comparison, we showed also the result of approximations with parameters τ_k^c and τ_k^s , $k = 1, \dots, p$ as the roots of $L_p^{(2q)}(x)$ (see Figure 3.3 and Theorem 3.4). In the figures, we call the latest as "non-optimal". We see the impact of optimizations on the accuracy of the rational approximations at $x = \pm 1$.

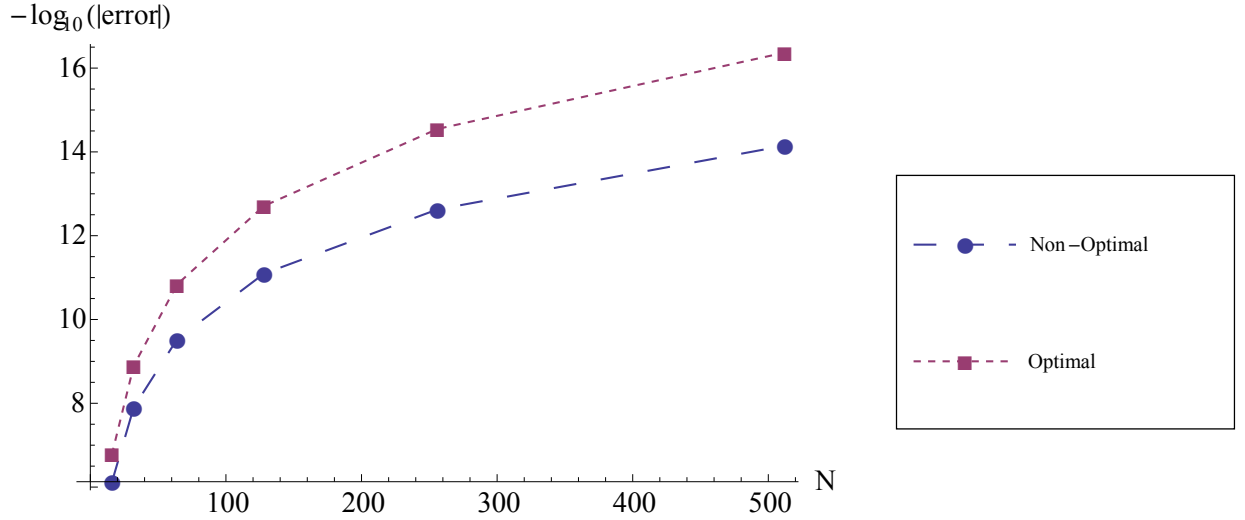


Figure 3.4: The values of $-\log_{10} \left(\max_{x=\pm 1} |R_{N,p}(f, \theta^c, \theta^s, x)| \right)$ and different N while approximating (2.25). In case of "non-optimal", parameters τ_k^c and τ_k^s , $k = 1, \dots, p$ are the roots of $L_p^{(2q)}(x)$. In case of "optimal", the parameters are chosen from Tables 3.1 and 3.2 for $p = 4$ and $q = 1$.

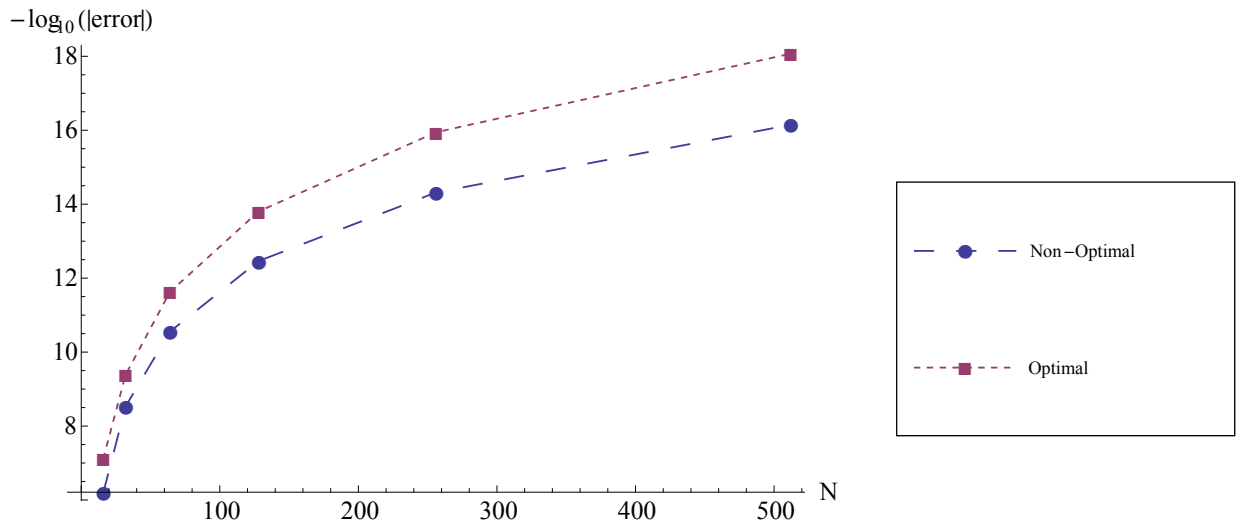


Figure 3.5: The values of $-\log_{10} \left(\max_{x=\pm 1} |R_{N,p}(f, \theta^c, \theta^s, x)| \right)$ for $p = 6$ and different N while approximating (2.25). In case of "non-optimal", parameters τ_k^c and τ_k^s , $k = 1, \dots, p$ are the roots of $L_p^{(2q)}(x)$. In case of "optimal", the parameters are chosen from Tables 3.1 and 3.2 for $p = 6$ and $q = 1$.

4. L_2 -convergence of the Modified Interpolations

In this section, we explore the convergence of the modified interpolation in the L_2 -norm. As we mentioned in Introduction, it is more convenient to rewrite the modified trigonometric system as follows (see (0.48))

$$\mathcal{H} = \{\varphi_n(x) : n \in \mathbb{Z}_+\}, \quad (4.1)$$

where

$$\begin{aligned} \varphi_0(x) &= \frac{1}{\sqrt{2}}, \\ \varphi_n(x) &= \frac{1}{2} \left((-1)^n e^{\frac{i\pi n x}{2}} + e^{-\frac{i\pi n x}{2}} \right), \quad n \in \mathbb{N}. \end{aligned} \quad (4.2)$$

Then, the truncated modified trigonometric series can be rewritten more compactly

$$M_N(f, x) = \sum_{n=0}^{2N} f_n^m \varphi_n(x), \quad (4.3)$$

where

$$f_n^m = \int_{-1}^1 f(x) \overline{\varphi_n(x)} dx. \quad (4.4)$$

Next lemma establishes connection between the modified discrete \check{f}_n^m (see (0.51)) and the continuous coefficients f_n^m .

Lemma 4.1 [13] *Assume that $f \in C^2[-1, 1]$ and $f'' \in BV[-1, 1]$. Then, the following identity holds*

$$\check{f}_n^m = f_n^m + \sum_{j=1}^{\infty} f_{n+(2N+1)2j}^m + (-1)^n \sum_{j=1}^{\infty} f_{-n+(2N+1)2j}^m, \quad n = 1, \dots, 2N. \quad (4.5)$$

Proof. From the pointwise convergence of the modified Fourier expansion (see Theorem 0.2 with $q = 0$), we have

$$f(x) = \sum_{j=0}^{\infty} f_j^m \varphi_j(x) = \sum_{r=0}^{\infty} \sum_{j=0}^{4N+1} f_{j+2r(2N+1)}^m \varphi_{j+2r(2N+1)}(x). \quad (4.6)$$

Taking into account that $\varphi_{j+2r(2N+1)}(x_k) = \varphi_j(x_k)$, we write

$$\check{f}_n^m = \sum_{r=0}^{\infty} \sum_{j=0}^{4N+1} f_{j+2r(2N+1)}^m \frac{2}{2N+1} \sum_{k=-N}^N \varphi_j(x_k) \overline{\varphi_n(x_k)}. \quad (4.7)$$

As we mentioned, the modified trigonometric system is discrete orthogonal for the grid $x_k = \frac{2k}{2N+1}$, $k = 0, \pm 1, \dots, \pm N$, which means that

$$\frac{2}{2N+1} \sum_{n=-N}^N \varphi_n(x_k) \overline{\varphi_m(x_k)} = \delta_{n,m}, \quad 0 \leq m, n \leq 2N \quad (4.8)$$

and

$$\frac{2}{2N+1} \sum_{n=0}^{2N} \varphi_n(x_k) \overline{\varphi_n(x_s)} = \delta_{k,s}, \quad |k|, |s| \leq N. \quad (4.9)$$

Moreover, it is easy to verify that for $j = 2N+1, \dots, 4N+1$

$$\frac{2}{2N+1} \sum_{k=-N}^N \varphi_j(x_k) \overline{\varphi_n(x_k)} = \begin{cases} 0, & n = 0 \\ (-1)^n \delta_{4N+2-n,j}, & 1 \leq n \leq 2N \end{cases} \quad (4.10)$$

All these together completes the proof due to (4.7). \square

We can rewrite Lemma 4.1 for coefficients f_n^s (see (0.4) and (0.57)) as follows.

Remark 4.1 *Assume that $f \in C^2[-1, 1]$ and $f'' \in BV[-1, 1]$. Then, the following identity holds*

$$\check{f}_n^s = f_n^s + \sum_{j \neq 0} f_{n+(2N+1)j}^s, \quad n = 1, \dots, N. \quad (4.11)$$

The next theorem describes the convergence of the modified interpolation (see (0.58)) in the L_2 -norm. Recall that for even functions, expansions and interpolations by the modified trigonometric system coincide with the classical expansions and interpolations. That is why, we formulate the convergence theorems only for odd functions on $[-1, 1]$.

Theorem 4.1 [13] *Let f be odd function on $[-1, 1]$. Assume that $f \in C^{2q+1}[-1, 1]$ and $f^{(2q+1)} \in BV[-1, 1]$, $q \geq 0$. Then, the following estimate holds*

$$\lim_{N \rightarrow \infty} N^{2q+\frac{3}{2}} \|r_N^q\|_{L_2} = |B_{2q+1}(f)| \frac{\sqrt{a(q)}}{\pi^{2q+2}}, \quad (4.12)$$

where

$$a(q) = \frac{1}{4q+3} + \int_0^1 \left(\sum_{s \neq 0} \frac{(-1)^s}{(2s+x)^{2q+2}} \right)^2 dx. \quad (4.13)$$

Proof. We can rewrite $r_N^q(f, x)$ (see (0.59)) for odd f as follows

$$r_N^q(f, x) = \sum_{n=1}^N (F_n^s - \check{F}_n^s) \sin \pi(n - \frac{1}{2})x + \sum_{n=N+1}^{\infty} F_n^s \sin \pi(n - \frac{1}{2})x. \quad (4.14)$$

Due to the orthonormality of the system functions of \mathcal{H} , we get

$$\|r_N^q\|_{L_2}^2 = \sum_{n=1}^N (F_n^s - \check{F}_n^s)^2 + \sum_{n=N+1}^{\infty} (F_n^s)^2. \quad (4.15)$$

Taking into account that function F obeys the first q derivative conditions (0.15), we derive the following asymptotic expansion of its modified Fourier coefficients by means of integration by parts

$$F_n^s = B_{2q+1}(f) \frac{(-1)^{n+1}}{(\pi(n - \frac{1}{2}))^{2q+2}} + o(n^{-2q-2}). \quad (4.16)$$

Then, application of Remark 4.1 leads to the following estimate for $n = 1, \dots, N$

$$\check{F}_n^s - F_n^s = B_{2q+1}(f) \frac{(-1)^{n+1}}{(\pi N)^{2q+2}} \sum_{j \neq 0} \frac{(-1)^j}{(2j + \frac{n}{N})^{2q+2}} + o(N^{-2q-2}). \quad (4.17)$$

Estimates (4.16) and (4.17), together with (4.15), complete the proof. \square

When $q = 0$, Theorem 4.1 shows convergence rate $O(N^{-\frac{3}{2}})$ in the L_2 -norm. The classical interpolation has convergence rate $O(N^{-\frac{1}{2}})$ in the L_2 -norm for odd functions on $[-1, 1]$ (see [33]). Hence, the improvement is by factor $O(N)$.

Let us consider the following odd function on $[-1, 1]$

$$f(x) = x^2 \sin x. \quad (4.18)$$

By $\varepsilon_1(N) = \|f(x) - \mathcal{I}_N^{\text{classic}}(f, x)\|_{L_2}$ and $\varepsilon_2(N) = \|f(x) - \mathcal{I}_N(f, x)\|_{L_2}$, we denote the L_2 -errors of the classic and modified interpolations, respectively.

Table 4.1 compares $\varepsilon_1(N)$ and $\varepsilon_2(N)$ for (4.18) and different values of N . Numerical results almost confirm the estimate of Theorem 4.1.

N	32	64	128	256	512	1024	2048
$\varepsilon_1(N)$	0.0978	0.0694	0.0492	0.0348	0.0246	0.0174	0.0123
$\varepsilon_2(N)$	0.0020	0.0007	0.0003	$9 \cdot 10^{-5}$	$3 \cdot 10^{-5}$	$1 \cdot 10^{-5}$	$4 \cdot 10^{-6}$

Table 4.1: L_2 -errors of the classical and modified interpolations for $N = 64, 128, \dots, 2048$ while approximating (4.18).

5. Pointwise Convergence of the Modified Interpolations

In this section, we investigate the pointwise convergence of the modified interpolations on $|x| < 1$ and the endpoints $x = \pm 1$. We need some auxiliary estimates for the proof of the main results. We will frequently use the properties of the following numbers

$$\alpha_{p,m} = \sum_{s=0}^p \binom{p}{s} (-1)^s s^m, \quad (5.1)$$

which are connected with the Stirling numbers of the second kind ([38]). In [23] it was verified that

$$\alpha_{p,m} = 0, \quad 0 \leq m < p, \quad \alpha_{p,p} = (-1)^p p!, \quad \alpha_{p,p+1} = (-1)^p \frac{p(p+1)!}{2}. \quad (5.2)$$

Let $\hat{c} = \{c_n\}$ be a sequence of complex numbers and $\Delta_n^p(\hat{c})$ be a finite differences

$$\Delta_n^p(\hat{c}) = \sum_{s=0}^p \binom{p}{s} c_{n+p-s}, \quad p \geq 0. \quad (5.3)$$

Let

$$Q_n(m) = \frac{(-1)^{n+1}}{(\pi(n - \frac{1}{2}))^{2m+2}}, \quad \hat{Q}(m) = \{Q_n(m)\}_{n=1}^{\infty}. \quad (5.4)$$

From (0.22) and asymptotic expansion (4.16), it follows that $Q_n(m)$ are the modified Fourier coefficients of the correction polynomial $Q_m(x)$. Then, denote

$$\check{Q}(m) = \{\check{Q}_n^s(m)\}_{n=0}^{2N}, \quad (5.5)$$

where $\check{Q}_n^s(m)$ are the discrete modified coefficients of $Q_m(x)$ (see (0.57)).

Lemma 5.1 [13] *For any $p \geq 0$ and $m \geq 0$, the following estimate holds*

$$\Delta_n^{2p}(Q(m)) = \frac{(-1)^{n+p+1} (2m+2p+1)!}{(\pi(n - \frac{1}{2}))^{2m+2} (n - \frac{1}{2})^{2p} (2m+1)!} + O(n^{-2m-2p-3}), \quad n \rightarrow \infty. \quad (5.6)$$

Proof. From definition of $\Delta_n^p(Q(m))$, we have

$$\begin{aligned}\Delta_n^{2p}(\hat{Q}(m)) &= \sum_{s=0}^{2p} \binom{2p}{s} Q_{n+p-s}(m) = \frac{(-1)^{n+p+1}}{(\pi(n-\frac{1}{2}))^{2m+2}} \sum_{s=0}^{2p} \frac{\binom{2p}{s} (-1)^k}{\left(1 + \frac{p-k}{n-\frac{1}{2}}\right)^{2m+2}} \\ &= \frac{(-1)^{n+p+1}}{(\pi(n-\frac{1}{2}))^{2m+2}} \sum_{s=0}^{\infty} \binom{s+2m+1}{2m+1} \frac{(-1)^s}{(n-\frac{1}{2})^s} \sum_{j=0}^s \binom{s}{j} (-1)^j p^{s-j} \alpha_{2p,j},\end{aligned}\tag{5.7}$$

where $\alpha_{2p,j}$ are defined by (5.1). This completes the proof in view of (5.2). \square

Lemma 5.2 [13] *For any $p \geq 0$ and $m \geq 0$, the following estimates hold*

$$\begin{aligned}\Delta_n^{2p}(\check{Q}(m) - \hat{Q}(m)) &= \frac{(-1)^{n+p+1}(2m+2p+1)!}{(\pi N)^{2m+2} N^{2p}(2m+1)!} \sum_{j \neq 0} \frac{(-1)^j}{\left(2j + \frac{n}{N}\right)^{2m+2p+2}} \\ &\quad + O(N^{-2m-2p-3}), \quad n = 1, \dots, N, \quad N \rightarrow \infty.\end{aligned}\tag{5.8}$$

Proof. According to Remark 4.1, we can write

$$\begin{aligned}\Delta_n^{2p}(\check{Q}(m) - \hat{Q}(m)) &= \sum_{j \neq 0} \Delta_{n+(2N+1)j}^{2p}(\hat{Q}(m)) \\ &= \frac{(-1)^{n+p+1}}{(\pi N)^{2m+2}} \sum_{k=0}^{2p} \binom{2p}{k} (-1)^k \sum_{j \neq 0} \frac{(-1)^j}{\left(2j + \frac{n}{N}\right)^{2m+2}} \frac{1}{\left(1 + \frac{j+p-k-\frac{1}{2}}{N(2j+\frac{n}{N})}\right)^{2m+2}} \\ &= \frac{(-1)^{n+p}}{(\pi N)^{2m+2}} \sum_{t=0}^{\infty} \frac{(-1)^t}{N^t} \binom{2m+1+t}{2m+1} \\ &\quad \times \sum_{s=0}^t \binom{t}{s} (-1)^s \alpha_{2p,s} \sum_{j \neq 0} \frac{(-1)^j (p+j-\frac{1}{2})^{t-s}}{\left(2j + \frac{n}{N}\right)^{2m+2}},\end{aligned}\tag{5.9}$$

where $\alpha_{2p,s}$ are defined by (5.1). This completes the proof in view of (5.2). \square

Lemma 5.3 [13] *For any $m \geq 0$ the following estimate holds*

$$\begin{aligned}\Delta_N^{2p}(\check{Q}(m)) &= \frac{(-1)^{N+p}(2m+2p+2)!}{(\pi N)^{2m+2} N^{2p+1}(2m+1)!} \sum_{j=-\infty}^{\infty} \frac{(-1)^j (j-\frac{1}{2})}{(2j+1)^{2m+2p+3}} \\ &\quad + O(N^{-2m-2p-4}), \quad N \rightarrow \infty.\end{aligned}\tag{5.10}$$

Proof. From Remark 4.1, we have

$$\begin{aligned}\Delta_N^{2p}(\check{Q}(m)) &= \sum_{j=-\infty}^{\infty} \Delta_{N+(2N+1)j}^{2p}(\hat{Q}(m)) \\ &= \frac{(-1)^{N+p+1}}{(\pi N)^{2m+2}} \sum_{t=0}^{\infty} \frac{(-1)^t}{N^t} \binom{2m+1+t}{2m+1} \sum_{s=0}^t \binom{t}{s} (-1)^s \alpha_{2p,s} \\ &\quad \times \sum_{j=-\infty}^{\infty} \frac{(-1)^j (p+j-\frac{1}{2})^{t-s}}{(2j+1)^{2m+2}},\end{aligned}\tag{5.11}$$

where $\alpha_{2p,s}$ are defined by (5.1). Taking into account that

$$\sum_{j=-\infty}^{\infty} \frac{(-1)^j}{(2j+1)^{2m+2}} = 0, \quad m = 0, 1, \dots \quad (5.12)$$

and identities (5.2), we complete the proof. \square

Next theorem demonstrates the pointwise convergence of the modified interpolation away from the endpoints.

Theorem 5.1 [13] *Let f be an odd function on $[-1, 1]$. Assume that $f \in C^{2q+3}[-1, 1]$ and $f^{(2q+3)} \in BV[-1, 1]$, $q \geq 0$. Then, the following estimate holds for $|x| < 1$*

$$r_N^q(f, x) = B_{2q+1}(f) \frac{(-1)^N}{N^{2q+3}} \frac{\pi |E_{2q+2}|}{2^{2q+5}(2q+1)!} \frac{\sin \pi(N + \frac{1}{2})x}{\cos^2 \frac{\pi x}{2}} + o(N^{-2q-3}), \quad N \rightarrow \infty, \quad (5.13)$$

where E_k is the k -th Euler number.

Proof. We put

$$f_{-n}^s = -f_{n+1}^s, \quad \check{f}_{-n}^s = -\check{f}_{n+1}^s, \quad (5.14)$$

to rewrite interpolation error (4.14) in a more convenient form

$$\begin{aligned} r_N^q(f, x) &= \frac{1}{2i} \sum_{n=-N+1}^N (F_n^s - \check{F}_n^s) e^{i\pi(n-\frac{1}{2})x} \\ &\quad + \frac{1}{2i} \sum_{n=N+1}^{\infty} F_n^s e^{i\pi(n-\frac{1}{2})x} \\ &\quad + \frac{1}{2i} \sum_{n=-\infty}^{-N} F_n^s e^{i\pi(n-\frac{1}{2})x}. \end{aligned} \quad (5.15)$$

We proceed by application of the Abel transformation and derive

$$\begin{aligned} r_N^q(f, x) &= \frac{1}{2(1 + \cos \pi x)} (\check{F}_{N+1}^s \sin \pi(N - \frac{1}{2})x - \check{F}_N^s \sin \pi(N + \frac{1}{2})x) \\ &\quad + \frac{1}{4(1 + \cos \pi x)^2} (\Delta_{N+1}^2(\check{F}^s) \sin \pi(N - \frac{1}{2}) - \Delta_N^2(\check{F}^s) \sin \pi(N + \frac{1}{2})) \\ &\quad + \frac{e^{-i\frac{\pi x}{2}}}{8(1 + \cos \pi x)^2} \left(\sum_{n=1}^N \Delta_n^4(F^s - \check{F}^s) e^{i\pi n x} + \sum_{n=N+1}^{\infty} \Delta_n^4(F^s) e^{i\pi n x} \right) \\ &\quad + \frac{e^{i\frac{\pi x}{2}}}{8(1 + \cos \pi x)^2} \left(\sum_{n=-N}^{-1} \Delta_n^4(F^s - \check{F}^s) e^{i\pi n x} + \sum_{n=-\infty}^{-N-1} \Delta_n^4(F^s) e^{i\pi n x} \right). \end{aligned} \quad (5.16)$$

Taking into account the following asymptotic expansion of the modified coefficients

$$F_n^s = \sum_{m=q}^{q+1} B_{2m+1}(f) Q_n(m) + o(n^{-2q-4}), \quad n \rightarrow \infty, \quad (5.17)$$

we get

$$\Delta_n^{2p}(F^s) = \sum_{m=q}^{q+1} B_{2m+1}(f) \Delta_n^{2p}(\hat{Q}(m)) + o(n^{-2q-4}), \quad n \rightarrow \infty. \quad (5.18)$$

Now, according to Lemma 5.1, we have

$$\Delta_n^4(F^s) = o(n^{-2q-4}), \quad (5.19)$$

and the infinite sums on the right-hand side of (5.16) are $o(N^{-2q-3})$. Again from (5.17), we write

$$\begin{aligned} \Delta_n^4(\check{F}^s - F^s) &= \sum_{m=q}^{q+1} B_{2m+1}(f) \Delta_n^4(\check{Q}(m) - \hat{Q}(m)) \\ &\quad + o(N^{-2q-3}), \end{aligned} \quad (5.20)$$

and from Lemma 5.2, we get

$$\Delta_n^4(F^s - \check{F}^s) = o(N^{-2q-4}), \quad n = \pm 1, \pm 2, \dots, \pm N. \quad (5.21)$$

Hence, the finite sums on the right-hand side of (5.16) are $o(N^{-2q-3})$.

Lemma 5.3 shows that

$$\Delta_N^2(\check{F}^s) = o(N^{-2q-3}), \quad (5.22)$$

and

$$\Delta_{N+1}^2(\check{F}^s) = o(N^{-2q-3}). \quad (5.23)$$

All these lead to the following estimate

$$\begin{aligned} r_N^q(f, x) &= \frac{1}{4 \cos^2 \frac{\pi x}{2}} \left(\check{F}_{N+1}^s \sin \pi \left(N - \frac{1}{2} \right) x - \check{F}_N^s \sin \pi \left(N + \frac{1}{2} \right) x \right) \\ &\quad + o(N^{-2q-3}). \end{aligned} \quad (5.24)$$

According to Lemma 5.3, we get

$$\begin{aligned} \check{F}_N^s &= B_{2q+1}(f) \frac{(-1)^N (2q+2)}{(\pi N)^{2q+2} N} \sum_{j=-\infty}^{\infty} \frac{(-1)^j (j - \frac{1}{2})}{(2j+1)^{2m+3}} \\ &\quad + o(N^{-2q-3}). \end{aligned} \quad (5.25)$$

From the other side

$$\check{F}_{N+1}^s = \frac{2}{2N+1} \sum_{k=-N}^N f(x_k) \sin \pi k = 0. \quad (5.26)$$

Hence,

$$r_N^q(f, x) = B_{2q+1}(f) \frac{(-1)^{N+1}(q+1) \sin \pi(N + \frac{1}{2})x}{2\pi^{2q+2}N^{2q+3} \cos^2 \frac{\pi x}{2}} \sum_{j=-\infty}^{+\infty} \frac{(-1)^j(j - \frac{1}{2})}{(2j+1)^{2q+3}} + o(N^{-2q-3}), \quad (5.27)$$

which completes the proof. \square

When $q = 0$, Theorem 5.1 implies the convergence rate $O(N^{-3})$ as $N \rightarrow \infty$ for an odd function. The classical interpolation (see [33]) has convergence rate $O(N^{-1})$ for the same grid. Hence, improvement is by factor $O(N^2)$ as $N \rightarrow \infty$.

Figures 5.1 and 5.2 show the graphs of the absolute errors of the classical and modified interpolations while approximating (4.18) for different values of N . Comparison shows that the modified interpolation is $O(N^2)$ times more accurate than the classical one, thus, confirming the estimate of Theorem 5.1.

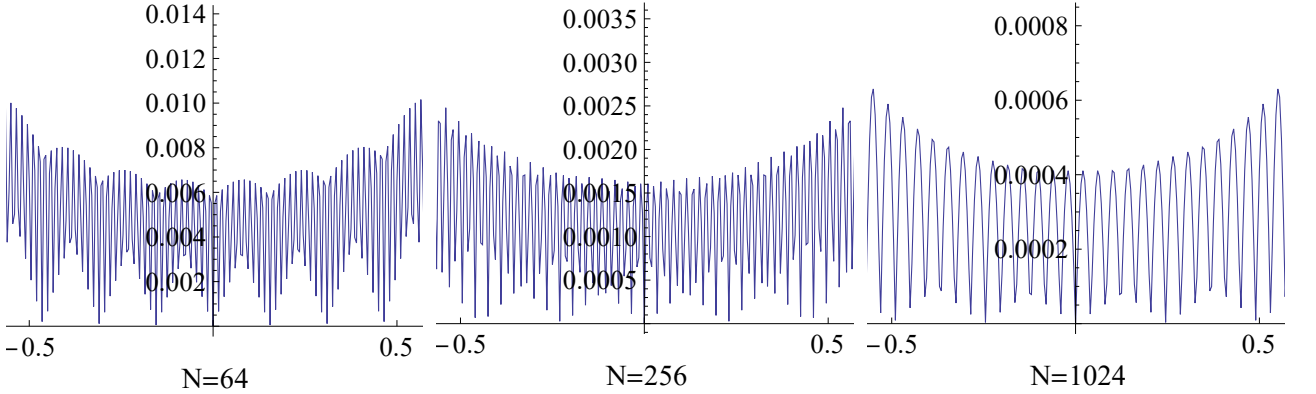


Figure 5.1: The graphs of the absolute errors while approximating (4.18) by the classical interpolation.

Next theorem explores the convergence of the modified interpolations at the endpoints $x = \pm 1$.

Theorem 5.2 [13] *Let f be an odd function on $[-1, 1]$. Assume that $f \in C^{2q+2}[-1, 1]$ and $f^{(2q+2)} \in BV[-1, 1]$, $q \geq 0$. Then, the following estimate holds*

$$r_N^q(f, \pm 1) = \pm B_{2q+1}(f) \frac{(-1)^{N+1}}{N^{2q+1}} \frac{|E_{2q}|}{2^{2q+1}\pi(2q+1)!} + o(N^{-2q-1}), \quad N \rightarrow \infty, \quad (5.28)$$

where E_k is the k -th Euler number.

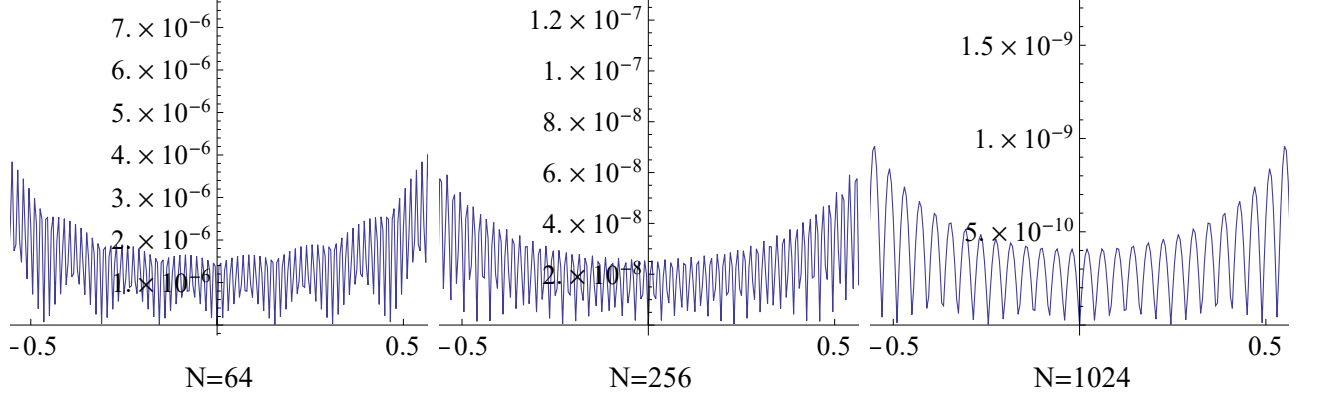


Figure 5.2: The graphs of the absolute errors while approximating (4.18) by the modified interpolation.

Proof. We use (4.14) and get

$$r_N^q(f, \pm 1) = \sum_{n=1}^N (F_n^s - \check{F}_n^s) (-1)^{n+1} + \sum_{n=N+1}^{\infty} F_n^s (-1)^{n+1}. \quad (5.29)$$

Taking into account the following asymptotic expansion of the modified Fourier coefficients

$$F_n^s = B_{2q+1}(f) Q_n(q) + o(n^{-2q-2}), \quad n \rightarrow \infty, \quad (5.30)$$

and applying Remark 4.1, we get for $n = 1, \dots, N$ and $N \rightarrow \infty$

$$\check{F}_n^s - F_n^s = \frac{B_{2q+1}(f) (-1)^{n+1}}{(\pi N)^{2q+2}} \sum_{j \neq 0} (-1)^j \frac{1}{(2j + \frac{n}{N})^{2q+2}} + o(N^{-2q-2}). \quad (5.31)$$

Equation (5.29), together with (5.30) and (5.31), implies

$$\begin{aligned} r_N^q(f, \pm 1) &= \pm B_{2q+1}(f) \frac{(-1)^N}{\pi^{2q+2} N^{2q+1}} \\ &\times \left(\frac{1}{2q+1} - \int_0^1 \sum_{j \neq 0} \frac{(-1)^j}{(2j+x)^{2q+2}} dx \right) + o(N^{-2q-1}), \end{aligned} \quad (5.32)$$

which completes the proof. \square

When $q = 0$, Theorem 5.2 shows convergence rate $O(1/N)$. In this case, as $f(1) \neq f(-1)$, the classical interpolation doesn't converge at the endpoints. Hence, the modified interpolations have better convergence rate at the endpoints and the improvement is by factor $O(N)$. We see that Figures 5.3 and 5.4 confirm this observation, where the graphs of the absolute errors while approximating (4.18) at the endpoints ± 1 by the classical and modified interpolations are presented.

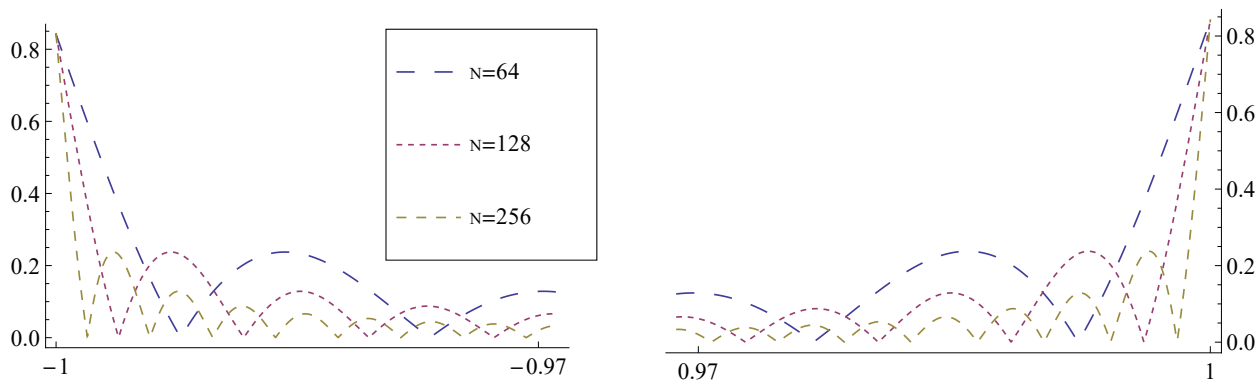


Figure 5.3: The graphs of the absolute errors at $x = \pm 1$ while approximating (4.18) by the classical interpolation.

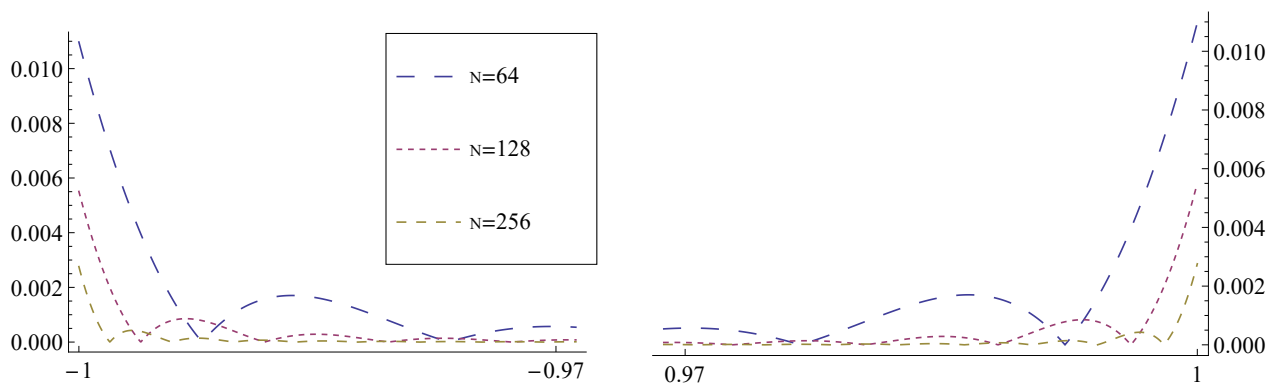


Figure 5.4: The graphs of the absolute errors at $x = \pm 1$ while approximating (4.18) by the modified interpolation.

Conclusion

This thesis is devoted to expansions and interpolations by the modified trigonometric system. It consists of five sections.

Sections 1-3 consider rational approximations by the modified trigonometric system. We call them as modified-trigonometric-rational (MTR-) approximations. The rational functions depend on some unknown parameters. We define those parameters differently. The first approach leads to the modified Fourier-Pade (MFP-) approximations (see (0.43) and (0.44)). The second approach is based on (0.45), where the values of parameters τ^c and τ^s are determined optimally to provide the best possible convergence rate. We call them as optimal MTR-approximations.

- Section 1 explores the convergence of the MFP-approximations:

- Theorem 1.1 explores the pointwise convergence for $|x| < 1$ and shows the exact constant of the asymptotic error. The convergence rate is $O(N^{-2q-2p-2})$ as $N \rightarrow \infty$. Compared to Theorem 0.4, the improvement in convergence rate is by factor $O(N^{2p})$.

- It is important to note that, in all theorems, for the modified expansions, we require less smoothness than for the rational approximations.

- Theorem 1.2 studies the convergence at $x = \pm 1$ and derives the exact constant of the asymptotic error. The convergence rate is $O(N^{2q+1})$ as $N \rightarrow \infty$. Comparison with Theorem 0.5 shows that the expansions by the modified Fourier system and the MFP-approximations have the same convergence rates at the endpoints $x = \pm 1$. However, comparison of the corresponding constants $h_{p,q}$ and $h_{0,q} = 1$ shows that the MFP-approximations are much more accurate than the classical expansions (see Table 1.1).

- Theorem 1.3 shows the exact constant of the asymptotic L_2 -error. Comparison of Theorems 0.3 and 1.3 shows that the classical expansions and the MFP-approximations have the same convergence rates $O(N^{-2q-3/2})$ in the L_2 -norm. However, comparison of the corresponding constants $c_{p,q}$ and $c_{0,q} = 1$ shows that the MFP-approximations are asymptotically

more accurate (see Table 1.3).

- Section 2 considers the pointwise convergence of the optimal MTR-approximations on $(-1, 1)$.

- Theorem 2.1 provides with general estimate on $|x| < 1$ proving that without optimal selection of parameters τ_k^c and τ_k^s , $k = 1, \dots, p$, the rational approximations have convergence rate $O(N^{-2q-p-2})$ as $N \rightarrow \infty$ if an approximated function has enough smoothness and obeys the first q derivative conditions (see (0.15)). Compared with the modified Fourier expansions (see Theorem 0.2), the improvement is by factor $O(N^p)$ as $N \rightarrow \infty$.

- Theorem 2.3 provides the optimal choice for parameters τ_k when $|x| < 1$ and p is odd. If $\tau_k^c = \tau_k^s$, $k = 1, \dots, p$ are the roots of the generalized Laguerre polynomial $L_p^{(2q+1)}(x)$ then, the rational approximations have convergence rate $O(N^{-2q-p-\lfloor \frac{p+1}{2} \rfloor - 2})$ with improvement by factor $O(N^{\lfloor \frac{p+1}{2} \rfloor})$ compared to non-optimal choice of parameters (Theorem 2.1). The improvement is by factor $O(N^{\lfloor \frac{p+1}{2} \rfloor + p})$ compared to the expansions by the modified Fourier system.

- Theorem 2.4 provides the optimal choice when $|x| < 1$ and p is even. It shows that the set of optimal parameters is wider compared to odd p . If polynomial (0.46) has only nonzero and real-valued roots $x = z_k$, $k = 1, \dots, p$ then, selection $\tau_k^s = \tau_k^c = z_k$ provides with better convergence rate $O(N^{-2q-p-\lfloor \frac{p}{2} \rfloor - 2})$ compared to the estimate of Theorem 2.1 and improvement is by factor $O(N^{\lfloor \frac{p}{2} \rfloor})$. Improvement is by factor $O(N^{\lfloor \frac{p}{2} \rfloor + p})$ compared to the expansions by the modified Fourier system. The problem is to find the values of c_1 in (0.46) for which it will have only real-valued and nonzero roots. In two cases it is obvious. When $c_1 = 0$, the roots of (0.46) coincide with the roots of $L_p^{(2q+1)}(x)$. When $c_1 = -1/(2q + p + 1)$, the roots coincide with the ones of $L_p^{(2q)}(x)$. In both cases all roots are positive.

- Theorem 2.5 explores the L_2 -error of the MTR-approximations without specifying the choice of the corresponding parameters. First, it derives the exact constant of the asymptotic L_2 -error. Then, parameters are selected such to minimize (numerically) the mentioned asymptotic constant. We call these approximations as L_2 -minimal MTR-approximations. Table 2.1 shows that the latests have better asymptotic L_2 -accuracy compared to the MFP-approximations.

- Section 3 considers the convergence of the optimal MTR-approximations at the endpoints $x = \pm 1$.

- Theorem 3.1 reveals the convergence rate of the MTR-approximations at $x = \pm 1$ without specifying parameters τ^c and τ^s . It derives the exact constant of the asymptotic error, which helps to determine the optimal values of parameters for better convergence. Theorem 3.1 shows the convergence rate $O(N^{-2q-1})$ as $N \rightarrow \infty$. Comparison with Theorem 0.2 shows the same convergence rate.

- Using the explicit form of the exact constant, Theorem 3.3 finds the optimal values of parameters for odd p for better convergence rate at $x = \pm 1$. It proves that the best accuracy could be achieved when parameters $\tau_k^s = \tau_k^c$ are the roots of the generalized Laguerre polynomial $L_p^{(2q)}(x)$. For that choice, the convergence rate is $O(N^{-2q - \lfloor \frac{p+1}{2} \rfloor - 1})$ and improvement is by factor $O(N^{\lfloor \frac{p+1}{2} \rfloor})$ compared to the modified Fourier expansions.

- When p is odd, the optimal choices for $|x| < 1$ and $x = \pm 1$ are different. The choice of polynomial $L_p^{(2q)}(x)$ will provide with the minimal uniform error on $[-1, 1]$, but for $|x| < 1$, the convergence rate will be worse by factor $O(N)$ compared to the optimal choice $L_p^{(2q+1)}(x)$.

- Theorem 3.4 outlines the set of optimal parameters for even p . It shows that the optimal choice is $\tau_k^c = \tau_k^s = z_k$, $k = 1, \dots, p$, where z_k are real-valued and non-zero roots of (0.47). It provides convergence rate $O(N^{-2q - \lfloor \frac{p}{2} \rfloor - 1})$ with improvement by factor $O(N^{\lfloor \frac{p}{2} \rfloor})$ compared to the modified Fourier expansions. Polynomial (0.47) has only real-valued and non-zero roots when $d_1 = 0$ or $d_1 = -1/(2q + p)$. For the first choice, the roots coincide with the ones of $L_p^{(2q)}(x)$ and for the second choice, with the roots of $L_p^{(2q-1)}(x)$. The choice of $L_p^{(2q)}(x)$ is better as it will provide with optimal approximations both for $|x| < 1$ and $x = \pm 1$.

- However, estimates of Theorem 3.4 allow to determine parameter d_1 for even more better convergence rate. Tables 3.1 and 3.2 show some values of d_1 and parameters τ_k that will provide with convergence rate $O(N^{-2q - \lfloor \frac{p}{2} \rfloor - 2})$ with improvement by factor $O(N)$. The problem is that the latest choice is not optimal for $|x| < 1$. It will give worse accuracy compared to the optimal selection for $|x| < 1$. Convergence rate will degrade by factor $O(N)$. As in case of odd p , a user of the algorithms must decide which choice will be more appropriate to select, the

best approximation on overall $[-1, 1]$ with worse accuracy on $|x| < 1$, or the best accuracy on the latest.

- The optimal values of parameters τ^c and τ^s depend only on p and q and are independent of f . It means that if functions f , g and $f + g$ have enough smoothness and obey the same derivative conditions, the optimal approach leads to linear rational approximations in the sense that

$$M_{N,p}(f + g, \theta^c, \theta^s, x) = M_{N,p}(f, \theta^c, \theta^s, x) + M_{N,p}(g, \theta^c, \theta^s, x) \quad (5.33)$$

with the same parameters θ^c and θ^s for all included functions.

• Sections 4 and 5 explore the convergence of the modified interpolation in different frameworks. Comparison with the classical interpolation, for the same uniform grid, confirms better convergence properties of the modified interpolations for odd functions on $[-1, 1]$ in all frameworks. Section 4 studies the L_2 -convergence of the modified interpolation. Section 5 explores the pointwise convergence of the modified interpolation.

- Theorem 4.1 reveals the convergence rate in the L_2 -norm. It presents the exact constant of the asymptotic error in the L_2 -norm. It shows that $r_N^q(f, x)$ (see (0.59)) is $O(N^{-2q-3/2})$ as $N \rightarrow \infty$. The modified interpolation has the same convergence rate as expansions by the modified trigonometric system (see Theorem 0.3).

- When $q = 0$, Theorem 4.1 shows convergence rate $O(N^{-\frac{3}{2}})$ in the L_2 -norm. The classical interpolation (see (0.60)) has convergence rate $O(N^{-\frac{1}{2}})$ in the L_2 -norm for odd functions on $[-1, 1]$. Hence, the improvement is by factor $O(N)$ for odd functions. Recall that for even functions on $[-1, 1]$, the modified interpolation is identical to the classical interpolation.

- Theorem 5.1 explores the pointwise convergence on $|x| < 1$ and derives the exact constant of the asymptotic error for a fixed $x \in (-1, 1)$. The convergence rate of r_N^q is $O(N^{-2q-3})$ which is better than the which is better than the convergence rate of the expansions by the modified trigonometric system and improvement is by factor $O(N)$ (see Theorem 0.4).

- When $q = 0$, Theorem 5.1 implies the convergence rate $O(N^{-3})$ as $N \rightarrow \infty$. The classical interpolation has convergence rate $O(N^{-1})$ for the same uniform grid on $[-1, 1]$. Hence, the improvement is by factor $O(N^2)$ for odd functions.

- Theorem 5.2 reveals the exact constant of the asymptotic error when $x = \pm 1$. It shows that the convergence rate of r_N^q is $O(N^{-2q-1})$ which is the same as for the convergence rate of the expansions by the modified trigonometric system.

- When $q = 0$, Theorem 5.2 shows convergence rate $O(1/N)$. In this case, as $f(1) \neq f(-1)$, the classical interpolation doesn't converge at the endpoints. Hence, the modified interpolations have better convergence rate at the endpoints with improvement by factor $O(N)$.

Notations

\hat{f} — a sequence $\{f_n\}_{n=1}^{\infty}$

f_n^c — modified Fourier cosine coefficient, see (0.4)

f_n^s — modified Fourier sine coefficient, see (0.4)

f_n^m — the modified Fourier coefficient, see (4.4)

\check{f}_n^c — the modified Fourier discrete cosine coefficient, see (0.56)

\check{f}_n^s — the modified Fourier discrete sine coefficient, see (0.57)

\check{f}_n^m — the modified Fourier discrete coefficient, see (0.51)

\check{f}_n — the classic Fourier discrete coefficient, see (0.60)

\mathcal{H} — the modified Fourier system

\mathcal{H}_{class} — the classic Fourier system

\mathcal{H}^* — the classic Fourier cosine system

$\mathcal{I}_N(f, x)$ — the modified interpolation, see (0.50)

$\mathcal{I}_N^q(f, x)$ — see (0.58)

$\mathcal{I}_N^{classic}(f, x)$ — the classic interpolation, see (0.60)

$M_N(f, x)$ — truncated modified Fourier series, see (0.3)

$M_N^q(f, x)$ — see (0.23)

$M_{N,p}(f, \theta^c, \theta^s, x)$ — the MTR-approximation, see (0.38)

$M_{N,p}^q(f, \theta^c, \theta^s, x)$ — see (0.40)

$R_N(f, x)$ — error of the modified Fourier expansion, see (0.30)

$R_N^q(f, x)$ — see (0.24)

$R_N^{\cos}(f, x)$ — cosine error of the modified Fourier expansion (see (0.31))

$R_N^{\sin}(f, x)$ — sine error of the modified Fourier expansion (see (0.32))

$R_{N,p}^{\cos}(f, \theta, x)$ — cosine error of the MTR-approximation (see (0.36))

$R_{N,p}^{\sin}(f, \theta, x)$ — sine error of the MTR-approximation (see (0.37))

$R_{N,p}(f, \theta^c, \theta^s, x)$ — error of the MTR-approximation (see (0.39))

$R_{N,p}^q(f, \theta^c, \theta^s, x)$ — see (0.41)

$r_N(f, x)$ — error of the modified interpolation (see (0.52))

$r_N^q(f, x)$ — see (0.59)

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