ԵՐԵՎԱՆԻ ՊԵՏԱԿԱՆ ՀԱՄԱԼՍԱՐԱՆ

Յուրի Արմենի Աշրաֆյան

Հակադարձ խնդիրներ որոշ դիֆերենցիալ օպերատորների համար

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Inverse problems for some differential operators

SYNOPSIS

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Ատենախոսության թեման հաստատվել է Երևանի Պետական Համալսարանում

Գիտական ղեկավար՝	ֆիզ.-մաթ. գիտ. դոկտոր
	Տ. Ն. Հարությունյան
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The thesis can be found at the YSU library.
The synopsis was sent on April 20, 2018.

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Overview

Relevance of the topic. In spectral theory of differential operators has been profoundly studied for the Sturm-Liouville problem. The first studies of such problems were performed by D. Bernoulli, J. d'Alembert, L. Euler in connection with the solution of the problem describing the vibration of a string. In a series of articles, dating from 1836-37, Sturm [24, 25] and Liouville [16] created a whole new subject in mathematical analysis. The Sturm-Liouville theory gave the first theorems on eigenvalue problems and as such it occupies a central place in the prehistory of functional analysis. The inverse problems as a separate branch of mathematics occurred from a work of V.A. Ambarzumian in 1929 ([1]), which in its turn was under the influence of quantum mechanics. Such problems often appear in mathematics, mechanics, physics, electronics, geophysics, meteorology and other branches of natural sciences. From the 1940's an excellent group of mathematicians: Borg, Levinson, Marchenko, Krein, Gelfand, Levitan, and others (see, e.g., [2, 12, 17, 18, 6, 13]), constructed a very rich theory of inverse problems connected with the equation of Sturm-Liuoville (Schrodinger), which used elegant results of different fields of mathematics, as the theory of operators, real and complex analysis, the theory of integral equations, etc., and developed them. Interest in this subject has been increasing permanently because of the appearance of new important applications, and nowadays the inverse problem theory develops intensively all over the world (see, e.g. journal "Inverse Problems").

The Dirac equation is a relativistic wave equation derived by British physicist Paul Dirac in 1928. It describes all spin-1/2 massive particles such as electrons and quarks (see [26]). It is consistent with both the principles of quantum mechanics and the theory of special relativity, and was the first theory to account fully for special relativity in the context of quantum mechanics. It is the system of partial differential equations, where unknown is 4-component vector-function. In the case of spherical-symmetric potential it reduces to ordinary differential system. From the end of 1950's and the beginning of 1960's also began the study of inverse problems for the Dirac equation (see, e.g. [20, 23, 28, 5]).

Goals. Find necessary and sufficient conditions for two sequences to be the set of eigenvalues and the set of norming constants of a Sturm-Liouville problem with in advance fixed boundary conditions. Generalize uniqueness theorems of Ambarzumyan and Marchenko. Investigate uniqueness theorems with the lowest eigenvalue.

Give the description of the isospectral Dirac operators on a finite interval in explicit form. Investigate the eigenvalues and eigenfunctions for Dirac operators with linear potential on whole and half axes. Give perturbations of Dirac operators with linear potential, when one add or subtract finite number of eigenvalues and scale the values of norming constants. Find formulae for eigenvalues' gradient on a finite interval and on half axis.

Research methods. Methods of theory of differential and integral equations, real and complex analysis.

Scientific novelty. All results are new and are the following:

1. The necessary and sufficient conditions for two sequences $\{\lambda_n^2\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ to be correspondingly the set of eigenvalues and the set of norming constants of a Sturm-Liouville problem with a real summable potential q and in advance fixed separated boundary conditions are found.

2. Connections of the set of norming constants and boundary parameters α and β are found.

3. A uniqueness theorem with the lowest eigenvalue $\mu_0(q, \alpha, \beta)$ for inverse Sturm-Liouville problem is proved.

4. The bounds for the lowest eigenvalue $\mu_0(q, \alpha, \beta)$ Sturm-Liouville problem are found.

5. It is shown, that there is a set of boundary conditions for which a generalization of Ambarzumyan's theorem is valid.

6. A new kind of uniqueness theorem, in some sense a generalization of Marchenko's theorem, conditioned by inequalities for inverse Sturm-Liouville problem is provided and proved. And other uniqueness theorems with inequalities are proved.

7. The description of all isospectral Dirac operators (on a finite interval), in explicit form and only in terms of the normalized eigenfunctions of the initial operator, is given.

8. The eigenvalues and eigenfunctions for Dirac operators with linear potential on whole and half axes are found in explicit form.

9. Perturbations of Dirac operators with linear potential (when one add or subtract finite number of eigenvalues and scale the values of norming constants) are constructed.

10. The concept of eigenvalues' gradient is given and formulae for this gradient are obtained on a finite interval and on an half axis.

11. The concept of eigenvalues' derivative with respect to canonical matrix-potential is introduced and shown how it is used for describing the isospectral Dirac operators or when finite number of spectral data is changed.

Theoretical and practical value. All the results and developed methods represent theoretical interest.

Approbation of results. Most of the results were reported in the following conferences:

in seminars of the chair of Differential Equations, "Annual sessions of the Armenian Mathematical Union (AMU)" - 2013, 2015 and 2016 (Yerevan, Armenia). "Second International Conference 'Mathematics in Armenia: Advances and Perspectives'" (Tsaghkadzor, Armenia, 2013). "IV annual conference of the Georgian Mathematical Union dedicated to academician V. Kupradze on his 110-th birthday anniversary" (Tbilisi, Batumi, Georgia, 2013). "Eight International Summer School on Geometry, Mechanics and Control", (Miraflores de la Sierra, Spain, 2014). "Inverse Problems: from Theory to Application" (Bristol, United Kingdom, 2014).

Publications. Main results of the thesis are published in 15 works (8 papers and 7 conference abstracts), which are listed at the end of references.

Structure and volume of the thesis. The thesis consists of preface, two chapters each has three sections, conclusion and bibliography with 68 items. Total number of pages is 87.

The content of the work

The Chapter 1 is devoted to Inverse Sturm-Liouville Problems (ISLP) on a finite interval.

In spectral theory of differential operators has been profoundly studied the Sturm-Liouville problem (SLP):

$$\ell y \equiv -y'' + q(x)y = \mu y, \qquad x \in (0, \pi), \tag{0.1}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \qquad \alpha \in (0, \pi], \tag{0.2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \qquad \beta \in [0, \pi), \tag{0.3}$$

which we denote by $L(q, \alpha, \beta)$, where $\mu \in \mathbb{C}$ is the spectral parameter and q is a real-valued, summable function $q \in L^1_{\mathbb{R}}[0, \pi]$. At the same time, by $L(q, \alpha, \beta)$ we also denote the self-adjoint operator, generated by this problem (0.1)–(0.3) in Hilbert space $L^2[0, \pi]$ (see, e.g. [21, 19, 14]).

It is known, that the spectrum of the operator $L(q, \alpha, \beta)$ is discrete and consists of countable real, simple eigenvalues (see, e.g. [19, 29, 8]), which we denote by $\mu_n = \mu_n(q, \alpha, \beta)$, $n = 0, 1, 2, \ldots$, emphasizing the dependence of μ_n on q, α and β. Let $\varphi(x, \mu) = \varphi(x, \mu, \alpha, q)$ and $\psi(x, \mu) = \psi(x, \mu, \beta, q)$ are the solutions of the equation (0.1), which satisfy the initial conditions

$$\varphi(0, \mu, \alpha, q) = \sin\alpha, \qquad \varphi'(0, \mu, \alpha, q) = -\cos\alpha,$$
$$\psi(\pi, \mu, \beta, q) = \sin\beta, \qquad \psi'(\pi, \mu, \beta, q) = -\cos\beta,$$

respectively. The functions $\varphi(x, \mu_n)$ and $\psi(x, \mu_n)$, $n \geq 0$, are the eigenfunctions, corresponding to the eigenvalue μ_n. The squares of the L^2-norm of these eigenfunctions:

$$a_n = a_n(q, \alpha, \beta) := \int_0^\pi \varphi^2(x, \mu_n)dx, \qquad n = 0, 1, 2, \ldots,$$
$$b_n = b_n(q, \alpha, \beta) := \int_0^\pi \psi^2(x, \mu_n)dx, \qquad n = 0, 1, 2, \ldots,$$

are called norming constants. Since all the eigenvalues are simple, there exist constants $\kappa_n = \kappa_n(q, \alpha, \beta) \neq 0$, $n = 0, 1, 2, \ldots$, such that

$$\varphi(x, \mu_n) = \kappa_n \; \psi(x, \mu_n).$$

The sequences $\{\mu_n\}_{n=0}^{\infty}$, $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{\kappa_n\}_{n=0}^{\infty}$ are called spectral data (besides these, there are other quantities, which are also called spectral data).

In general, the inverse spectral problem is to reconstruct the operator by some spectral data. Such problems often appear in mathematics, mechanics, physics, electronics, geophysics, meteorology and other branches of natural sciences. Interest in this subject has been increasing permanently because of the appearance of new important applications, and nowadays the inverse problem theory develops intensively all over the world.

In the cases $\sin\alpha \neq 0$ or/and $\sin\beta \neq 0$, often as norming constants is being considered the quantities $\tilde{a}_n = \dfrac{a_n}{\sin^2\alpha}$ and $\tilde{b}_n = \dfrac{b_n}{\sin^2\beta}$.

In Section 1.2 we describe the necessary and sufficient conditions for two sequences $\{\mu_n\}_{n=0}^{\infty} = \{\lambda_n^2\}_{n=0}^{\infty}$ and $\{\tilde{a}_n\}_{n=0}^{\infty}$ (the case $\sin\alpha \neq 0$) to be correspondingly the set of eigenvalues and the set of norming constants of a Sturm-Liouville problem with real summable potential q and in advance fixed separated boundary conditions. More precisely, we proof the following:

Theorem 1 *For a real increasing sequence $\{\lambda_n^2\}_{n=0}^{\infty}$ and a positive sequence $\{\tilde{a}_n\}_{n=0}^{\infty}$ to be spectral data for boundary-value problem $L(q,\alpha,\beta)$ with a $q \in L^1_{\mathbb{R}}[0,\pi]$ and fixed $\alpha, \beta \in (0,\pi)$ it is necessary and sufficient that the following relations hold:*

1) the sequence $\{\lambda_n\}_{n=0}^{\infty}$ has asymptotic form

$$\lambda_n = n + \frac{\omega}{n} + l_n,$$

where $\omega = const$, $l_n = o\left(\frac{1}{n}\right)$, when $n \to \infty$, and the function $l(\cdot)$, defined by formula $l(x) = \sum_{n=1}^{\infty} l_n \sin nx$, is absolutely continuous on arbitrary segment $[a,b] \subset (0, 2\pi)$,

2) the sequence $\{\tilde{a}_n\}_{n=0}^{\infty}$ has asymptotic form

$$\tilde{a}_n = \frac{\pi}{2} + s_n,$$

where $s_n = o\left(\frac{1}{n}\right)$, when $n \to \infty$, and the function $s(\cdot)$, defined by formula $s(x) = \sum_{n=1}^{\infty} s_n \cos nx$, is absolutely continuous on arbitrary segment $[a,b] \subset (0, 2\pi)$,

3)

$$\frac{1}{\tilde{a}_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty}\left(\frac{1}{\tilde{a}_n} - \frac{2}{\pi}\right) = \cot\alpha,$$

4)

$$\frac{\tilde{a}_0}{\left(\pi\prod_{k=1}^{\infty}\frac{\mu_k-\mu_0}{k^2}\right)^2}-\frac{1}{\pi}+\sum_{n=1}^{\infty}\left(\frac{\tilde{a}_n n^4}{\left(\pi[\mu_0-\mu_n]\prod_{k=1,k\neq n}^{\infty}\frac{\mu_k-\mu_n}{k^2}\right)^2}-\frac{2}{\pi}\right)=-\cot\beta.$$

An intensive development of the spectral theory for various classes of differential and integral operators took place in the 20-th century.

Historically, the first work in the theory of inverse spectral problems for Sturm-Liouville operators is due to Ambarzumyan [1]. Consider SLP generated by differential equation (0.1) with Neumann boundary conditions

$$y'(0)=0,$$
$$y'(\pi)=0,$$

i.e. $L(q,\pi/2,\pi/2)$. It is easy to calculate, that if $q(x)\equiv 0$, then the eigenvalues of the problem $L(0,\pi/2,\pi/2)$ are $\mu_n=n^2$, $n\geq 0$. He proved the inverse assertion, i.e. if the eigenvalues of the problem $L(q,\pi/2,\pi/2)$ are n^2, then the potential $q\equiv 0$.

Swedish mathematician Borg [2] was the first who paid attention to the importance of Ambarzumyan's result. Borg showed that in general for the boundary-value problem $L(q,\alpha,\beta)$ additional information is required in order to reconstruct the operator uniquely. Therefore, the Ambarzumyan's result was an exception from the rule and for a given spectrum, there exist infinitely many triples (q,α,β), such that the corresponding problems $L(q,\alpha,\beta)$ have the same spectrum. In the same work he showed that two spectra is sufficient for the unique determination of the operator.

The next step in the development of the classical inverse Sturm-Liouville problem was taken by N. Levinson [12]. Levinson, in 1949, proved, that in the class of even[1] Sturm-Liouville operators the spectrum $\{\mu_n\}_{n=0}^{\infty}$ uniquely determines the potential $q(x)$ and the parameter α.

In 1950 V. Marchenko [17], using transformation (transmutation) operators, proved that if two Sturm-Liouville problems on the half-line,

$$-y''+q_1(x)y=\mu y,\qquad y'(0)-h_1y(0)=0,$$
$$-y''+q_2(x)y=\mu y,\qquad y'(0)-h_2y(0)=0$$

[1]Sturm-Liouvile operator $L(q,\alpha,\beta)$ is called even, if $q(x)=q(\pi-x)$ and $\alpha+\beta=\pi$.

have the same spectral function, then $q_1(x) = q_2(x)$ and $h_1 = h_2$.

In Section 1.3 we bring various formulations of famous uniqueness theorems, then we give some new statements of uniqueness theorems. A uniqueness theorem with the lowest eigenvalue for Sturm-Liouville problems with arbitrary self-adjoint boundary conditions is proved.

Theorem 2 *Let q, $q_0 \in L^1_{\mathbb{R}}(0,\pi)$ and $\hat{q} = q - q_0$. If*

$$\mu_0(q) - \mu_0(q_0) = \operatorname{ess\,inf} \hat{q} \quad \textit{or} \quad \mu_0(q) - \mu_0(q_0) = \operatorname{ess\,sup} \hat{q},$$

then $q(x) = q_0(x) + \mu_0(q) - \mu_0(q_0)$ a.e. on $(0,\pi)$.

As a corollary of this Theorem, we find bounds for the lowest eigenvalue $\mu_0(q,\alpha,\beta)$.

Theorem 3 *Let $\alpha \in (0,\pi]$ and $\beta \in [0,\pi)$ and $q \in L^1_{\mathbb{R}}(0,\pi)$. The lowest eigenvalue $\mu_0(q,\alpha,\beta)$ has the property*

$$\operatorname{ess\,inf} q(x) + \mu_0(0,\alpha,\beta) \le \mu_0(q,\alpha,\beta) \le \operatorname{ess\,sup} q(x) + \mu_0(0,\alpha,\beta).$$

After that, we give a new proof of the famous generalization of Ambarzumyan's theorem with one spectrum (see [9, 11]).

Theorem 4 *Let $q' \in L^2_{\mathbb{R}}(0,\pi)$.*

If

$$\mu_n(q,\alpha,\pi-\alpha) = \mu_n(0,\alpha,\pi-\alpha),$$

for all $n \ge 0$, then $q(x) \equiv 0$.

After that a uniqueness theorem similar to Marchenko's theorem conditioned by inequality.

Theorem 5 *Let $q' \in L^2_{\mathbb{R}}(0,\pi)$. If*

$$\mu_n(q,\alpha_0,\beta) = \mu_n(q_0,\alpha_0,\beta_0),$$
$$\tilde{a}_n(q,\alpha_0,\beta) \ge \tilde{a}_n(q_0,\alpha_0,\beta_0),$$

for all $n \ge 0$, then $\beta = \beta_0$ and $q(x) \equiv q_0(x)$.

Furthermore, other uniqueness theorems conditioned by inequalities are proved (for $\kappa_n, \tilde{b}_n$ and $\varphi(\pi,\mu_n), \psi(0,\mu_n)$), e.g.

Theorem 6 *Let $q' \in L^2_{\mathbb{R}}(0,\pi)$. If*

$$\mu_n(q,\alpha_0,\beta) = \mu_n(q_0,\alpha_0,\beta_0),$$
$$|\kappa_n(q,\alpha_0,\beta)| \geq |\kappa_n(q_0,\alpha_0,\beta_0)|,$$

for all $n \geq 0$, then $\beta = \beta_0$ and $q(x) \equiv q_0(x)$.

The Chapter 2 is devoted to the direct and inverse spectral theory for Canonical Dirac System.

Let p and q are real-valued, summable on $[0,\pi]$ functions, $p,q \in L^1_{\mathbb{R}}[0,\pi]$. By $L(p,q,\alpha,\beta) = L(\Omega,\alpha,\beta)$ we denote the boundary-value problem for canonical Dirac system (see [27, 5, 14, 4, 19]):

$$\ell y \equiv \Big\{B\frac{d}{dx} + \Omega(x)\Big\}y = \lambda y, \quad x \in (0,\pi), \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{0.4}$$

$$y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0, \quad \alpha \in \Big(-\frac{\pi}{2},\frac{\pi}{2}\Big], \tag{0.5}$$

$$y_1(\pi)\cos\beta + y_2(\pi)\sin\beta = 0, \quad \beta \in \Big(-\frac{\pi}{2},\frac{\pi}{2}\Big], \tag{0.6}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}.$$

Matrix-function $\Omega(\cdot)$ is usually called potential function, and λ is a complex (spectral) parameter, $\lambda \in \mathbb{C}$. By the same $L(p,q,\alpha,\beta)$ we also denote a self-adjoint operator, generated by differential expression ℓ in Hilbert space of two component vector-functions $L^2([0,\pi];\mathbb{C}^2)$ on the domain

$$D = \Big\{y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; y_k \in AC[0,\pi], (\ell y)_k \in L^2[0,\pi], k = 1,2;$$
$$y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0,\ y_1(\pi)\cos\beta + y_2(\pi)\sin\beta = 0\Big\}$$

where $AC[0,\pi]$ is the set of absolutely continuous functions on $[0,\pi]$ (see, e.g. [21, 15]).

Two Dirac (or Sturm-Liouville) operators are said to be isospectral, if they have the same spectrum. The problem of description of all isospectral Sturm-Liouville operators was suggested and solved (for $q \in L^2_{\mathbb{R}}[0,\pi]$) by E. Trubowitz and coauthors in the series of works [10, 9, 3, 22].

For Dirac operators the description of all isospectral operators is given by Harutyunyan in [7]. That description has a "recurrent" form, i.e. at first only one norming constant is being changed, while the others stay unchanged, and obtained a new operator which has the same spectrum and the same norming constants except one. Then changing successively each norming constants, all isospectral operators were inferred, which have the given spectum. Note, that each operator is being obtained from the previous operator. This approach Harutyunyan calles "recurrent" description.

In Section 2.1 we give the description of all self-adjoint regular Dirac operators for with potential functions $p, q \in L^2_{\mathbb{R}}[0,\pi]$, on $[0,\pi]$, with the same spectrum, in explicit form, i.e. only in terms of normalized eigenfunctions of the initial operator $L(\Omega,\alpha,0)$ and a given sequence from l^2. With this aim we set $T = \{t_k\}_{k\in\mathbb{Z}} \in l^2$ and by $S(x,T)$ denote square matrix

$$S(x) = \left(\delta_{ij} + (e^{t_j} - 1)\int_0^x h_i^*(s)h_j(s)ds\right)_{i,j\in\mathbb{Z}}$$

where δ_{ij} is a Kronecker symbol, $h_n(x) = (h_{n_1}(x), h_{n_2}(x))^*$ are normalized eigenfunctions and $*$ is the sign of transponation. By $S_p^{(k)}(x,T)$ we denote a matrix, which is obtained from the matrix $S(x,T)$, when we replace $k-th$ column of $S(x,T)$ by $\{-(e^{t_k}-1)h_{k_p}(x)\}_{k\in\mathbb{Z}}$ column, $p = 1,2$. Now we can formulate our result as follow.

Theorem 7 *Let $T = \{t_k\}_{k\in\mathbb{Z}} \in l^2$ and $p, q \in L^2_{\mathbb{R}}[0,\pi]$. Then the isospectral operator $L(p(T), q(T), \alpha, 0)$, corresponding to T, is generated by potential, which is defined by formula*

$$\Omega(x,T) = \Omega(x) + G(x,x,T)B - BG(x,x,T) = \begin{pmatrix} p(x,T) & q(x,T) \\ q(x,T) & -p(x,T) \end{pmatrix},$$

where

$$G(x,x,T) = \frac{1}{\det S(x,T)}\sum_{k\in\mathbb{Z}}\begin{pmatrix} \det S_1^{(k)}(x,T) \\ \det S_2^{(k)}(x,T) \end{pmatrix} h_k^*(x).$$

In addition, for $p(x,T)$ and $q(x,T)$ we get an explicit representations:

$$p(x,T) = p(x) - \frac{1}{\det S(x,T)} \sum_{k\in\mathbb{Z}} \sum_{p=1}^{2} \det S_p^{(k)}(x,T) h_{k_{(3-p)}}(x),$$

$$q(x,T) = q(x) + \frac{1}{\det S(x,T)} \sum_{k\in\mathbb{Z}} \sum_{p=1}^{2} (-1)^{p-1} \det S_p^{(k)}(x,T) h_{k_p}(x).$$

In Section 2.2 we consider singular Dirac operators on whole and half axes with potential functions $p, q \in L^1_{\mathbb{R},loc}$. By $L(p,q)$ we denote a self-adjoint operator (see [21]) on whole axis, generated by differential expression ℓ (see (0.4)) in Hilbert space of two-component vector-functions $L^2((-\infty,\infty);\mathbb{C}^2)$ on the domain

$$D = \Big\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; y_k \in L^2(-\infty,\infty) \cap AC(-\infty,\infty);$$

$$(\ell y)_k \in L^2(-\infty,\infty), k = 1,2 \Big\},$$

where $AC(-\infty,\infty) = AC(\mathbb{R})$ is the set of functions, which are absolutely continuous on each finite segment $[a,b] \subset (-\infty,\infty), -\infty < a < b < \infty$. We assume that this operator has purely discrete spectrum, which consists of simple, real eigenvalues, which we denote by $\lambda_n(p,q)$ and enumerate in increasing order.

For $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$, by $L(p,q,\alpha)$ we denote a self-adjoint operator on half axis, generated by differential expression ℓ (see (0.4)) in Hilbert space of two component vector-functions $L^2((0,\infty);\mathbb{C}^2)$ on the domain

$$D_\alpha = \Big\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; y_k \in L^2(0,\infty) \cap AC(0,\infty);$$

$$(\ell y)_k \in L^2(0,\infty), k = 1,2;\ y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0 \Big\},$$

where $AC(0,\infty)$ is the set of functions, which are absolutely continuous on each finite segment $[a,b] \subset (0,\infty), 0 < a < b < \infty$. Here we also assume that this operator has purely discrete spectrum, which consists of simple, real eigenvalues, which we denote by $\lambda_n(p,q,\alpha)$ and enumerate in increasing order.

We prove that canonical Dirac expression with linear potential

$$\Omega_0(x) = \sigma_3 \cdot x = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}$$

generates operators on whole and half axises, for which we can find the eigenvalues and eigenfunctions in explicit form. Precisely, if we take the system of Chebyshev-Hermite orthonormal functions

$$\varphi_n(x) = \frac{e^{-\frac{x^2}{2}} H_n(x)}{\sqrt{2^n n! \sqrt{\pi}}}, \qquad n = 0, 1, 2, \ldots,$$

where

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}.$$

Then for Dirac operator $L(0, x)$ the vector-functions

$$U_{-n}(x) = \begin{pmatrix} -\varphi_{n-1}(x) \\ \varphi_n(x) \end{pmatrix}, \qquad U_0(x) = \begin{pmatrix} 0 \\ \varphi_0(x) \end{pmatrix}, \qquad U_n(x) = \begin{pmatrix} \varphi_{n-1}(x) \\ \varphi_n(x) \end{pmatrix},$$

for $n = 1, 2, \ldots$, are eigenfunctions corresponding to eigenvalues $\lambda_{-n} = -\sqrt{2n}, \lambda_0 = 0, \lambda_n = \sqrt{2n}$.

The spectral function of an operator $L(0, x, 0)$ is defined as [5, 14]

$$\rho(\lambda) = \begin{cases} \sum_{0<\lambda_n \leq \lambda} a_n^{-1}, & \lambda > 0, \\ -\sum_{\lambda<\lambda_n \leq 0} a_n^{-1}, & \lambda < 0, \end{cases}$$

and $\rho(0) = 0$, i.e. $\rho(\lambda)$ is left-continuous, step function with jumps in points $\lambda = \lambda_n$ equals a_n^{-1}.

We construct perturbations of these operators with in advance partially given spectrum, i.e. we answer the questions, what will happen with the potential $\Omega_0(x)$ if we add or subtract finite number of eigenvalues and change the values of norming constants. For instance, if we extract one eigenvalue, e.g. $\lambda_0(0, x, 0)$ we will get the following theorem.

Theorem 8 *Let $\rho(\lambda)$ is a spectral function of the operator $L(0, x, 0)$. Then the function $\tilde{\rho}(\lambda)$, defined by relation*

$$\tilde{\rho}(\lambda) = \begin{cases} \rho(\lambda), & \lambda \leq \lambda_0, \\ \rho(\lambda) - a_0^{-1}, & \lambda > \lambda_0, \end{cases}$$

where $a_0 = \sqrt{\pi}/2$, i.e.

$$d\tilde{\rho}(\lambda) = d\rho(\lambda) - \frac{1}{a_0}\delta(\lambda - \lambda_0) d\lambda$$

where $\delta(\lambda)$ is Dirac δ-function, is also spectral. Moreover, there exists unique self-adjoint canonical Dirac operator $\tilde{L}$ generated by the differential expression $\tilde{l} = B\frac{d}{dx} + \tilde{\Omega}(x)$ and the boundary condition $y_1(0) = 0$, for which $\tilde{\rho}(\lambda)$ is spectral function. Wherein, the potential function $\tilde{\Omega}(x)$ is represented by the following formula

$$\tilde{\Omega}(x) = \begin{pmatrix} 0 & x - \dfrac{e^{-x^2}}{a_0 - \int_0^x e^{-s^2} ds} \\ x - \dfrac{e^{-x^2}}{a_0 - \int_0^x e^{-s^2} ds} & 0 \end{pmatrix}$$

and for the eigenfunctions the following formulae hold

$$\tilde{V}_n(x) = \begin{pmatrix} V_{n,1}(x) \\ V_{n,2}(x) + \dfrac{e^{-\frac{x^2}{2}} \int_0^x e^{-\frac{s^2}{2}} V_{n,2}(s) ds}{a_0 - \int_0^x e^{-s^2} ds} \end{pmatrix}, \qquad n \in \mathbb{Z}\backslash\{0\}.$$

Section 2.3 is devoted to Dirac operators, which have discrete spectrum. The concept of eigenvalues' gradient is given (for both $L(p,q,\alpha,\beta)$ and $L(p,q,\alpha)$)

$$grad\lambda_n = \left(\frac{\partial\lambda_n}{\partial\alpha}, \frac{\partial\lambda_n}{\partial\beta}, \frac{\partial\lambda_n}{\partial p(x)}, \frac{\partial\lambda_n}{\partial q(x)}\right).$$

and formulae for this gradients are obtained in terms of normalized eigenfunctions.

Theorem 9 *Let λ_n and $h_n(x)$ are eigenvalues and normalized eigenfunctions of the problem $L(p,q,\alpha,\beta)$, correspondingly. Then the following relations are valid:*

$$\frac{\partial\lambda_n(\alpha,\beta,p,q)}{\partial\alpha} = -|h_n(0)|^2,$$
$$\frac{\partial\lambda_n(\alpha,\beta,p,q)}{\partial\beta} = |h_n(\pi)|^2,$$
$$\frac{\partial\lambda_n(\alpha,\beta,p,q)}{\partial p(x)} = |h_{n_1}(x)|^2 - |h_{n_2}(x)|^2,$$
$$\frac{\partial\lambda_n(\alpha,\beta,p,q)}{\partial q(x)} = 2h_{n_1}(x) \cdot h_{n_2}(x).$$

Similar formulae for operator $L(p,q,\alpha)$.

The concept of eigenvalues' derivative with respect to a canonical matrix-potential is introduced:

$$\frac{\partial \lambda_n}{\partial \Omega(x)} := \begin{pmatrix} \dfrac{\partial \lambda_n}{\partial p(x)} & \dfrac{\partial \lambda_n}{\partial q(x)} \\ \dfrac{\partial \lambda_n}{\partial q(x)} & -\dfrac{\partial \lambda_n}{\partial p(x)} \end{pmatrix}$$

and shown how it is used to describe the isospectral operators or when finite number of spectral data is changed.

Bibliography

[1] V. Ambarzumyan. *Über eine frage der eigenwertsththeori. Zeitschrift für Physik*, 53:690–695, 1929.

[2] G. Borg. *Eine Umkehrung der* Sturm–Liouvilleschen *Eigenwertaufgabe. Acta Math.*, 78:1–96, 1946.

[3] B. Dahlberg and E. Trubowitz. *The inverse Sturm-Liouville problem,* III. *Com. Pure and Appl. Math.*, 37:255–267, 1984.

[4] M. Gasymov and T. Dzhabiev. *Determination of the system of Dirac differential equations from two spectra. Proceedings of the Summer School in the Spectral Theory of Operators and the Theory of Group Representation (Baku, 1968), Izdat. Elm, Baku*, pages 46–71, 1975. (in Russian).

[5] M. Gasymov and B. Levitan. *Dokl. Akad. Nauk SSSR*, 167(5):967–970, 1966. (in Russian).

[6] I. Gelfand and B. Levitan. *On the Determination of a Differential Equation from its Spectral Function. Izv. Akad. Nauk. SSSR., Ser Math.*, 15:253–304, 1951.

[7] T. Harutyunyan. *Isospectral Dirac operators. Izv. Nats. Akad. Nauk Armenii Mat.*, 29(2):3–14, 1994. (in Russian, English version: J. Contemp. Math. Anal., 29(2, 1–10)).

[8] T. Harutyunyan. *The Sturm–Liouville boundary-value problem.* Yerevan State University, Yerevan, 2014. (in Armenian).

[9] E. Isaacson, H. McKean, and E. Trubowitz. *The inverse Sturm-Liouville problem,* II. *Com. Pure and Appl. Math.*, 37:1–11, 1984.

[10] E. Isaacson and E. Trubowitz. *The inverse Sturm-Liouville problem,* I. *Com. Pure and Appl. Math.*, 36:767–783, 1983.

[11] E. Korotyaev and D. Chelkak. *The inverse Sturm-Liouville problem with mixed boundary conditions. Algebra i Analiz*, 21(5):114–137, 2009.

[12] N. Levinson. *The inverse Sturm–Liouville problem. Mat. Tidsskr., B.*, pages 25–30, 1949.

[13] B. Levitan. *Inverse Sturm–Liouville problems.* Nauka, Moskva, 1984. (English transl., VSP, Zeist, 1987).

[14] B. Levitan and I. Sargsyan. *Introduction to spectral theory.* Nauka, Moskva, 1970. (in Russian).

[15] B. Levitan and I. Sargsyan. *Sturm-Liouville and Dirac operators.* Nauka, Moskva, 1988. (in Russian).

[16] J. Liouville. *Mémorie sur le développement des functions ou parites de functions en séries dont les divers termes sont assujettis à satisfaire à une même équation differentielles du second ordre contenant un paramétre variable. J. de Math*, 1:253–265, 1836.

[17] V. Marchenko. *Concerning the theory of a differential operator of the second order. Doklady Akad. Nauk., SSSR*, 72:457–460, 1950. (in Russian).

[18] V. Marchenko. *Concerning the theory of a differential operator of the second order. Trudy Moskov. Mat. Obshch.*, 1:327–420, 1952. (in Russian).

[19] V. Marchenko. *Sturm–Liouville operators and its applications.* Naukova Dumka, Kiiv, 1977. (in Russian).

[20] H. Moses. *The inverse problem for the Dirac system. Bull. Amer. Phys. Soc.*, 4:32–40, 1957.

[21] M. Naimark. *Linear differential operators.* Nauka, Moskva, 1969. (in Russian).

[22] J. Poschel and E. Trubowitz. *Inverse spectral theory.* New-York, Academic Press, 1987.

[23] F. Prats and J. Toll. *Phys. Rev.*, 113(1):363, 1959.

[24] C. Sturm. *Mémoire sur une classe d'Équations à différences partielles. J. de Math*, 1:373–444, 1836.

[25] C. Sturm. *Sur les équations differentielle linéares du second ordre. J. de Math*, 1:106–186, 1836.

[26] B. Thaller. *The Dirac Equation.* 1992.

[27] E. Titchmarsh. *Some eigenfunction expantion formulae. Proc. London Math. Soc.*, 11(3):159–168, 1961.

[28] M. Verde. *Nuclear Phys.*, 9:255, 1958.

[29] V. Yurko. *An introductions to the theory of inverse spectral problems.* Moskva, Fizmatlit, 2007. (in Russian).

The author's publications on the topic of the thesis

Articles

1. Ashrafyan Yu.A., *A remark on Ambarzumian's theorem.* Results in Mathematics, Vol. 79, Issue 1, No. 36, pp. 1-3, 2018.

2. Ashrafyan Yu.A., Harutyunyan T.N., *Inverse Sturm-Liouville problems with summable potential.* Mathematical Inverse Problems, Vol. 5, No. 1, pp. 35-45, 2018.

3. Ashrafyan Yu.A., *A new kind of uniqueness theorems for inverse Sturm-Liouville problems.* Boundary Value Problems, Vol. 2017, No. 79, pp. 1-8, 2017.

4. Harutyunyan T.N., Ashrafyan Yu.A., *Gradient of eigenvalues of Dirac operators and its applications.* Mathematical Inverse Problems, Vol. 4, No. 1, pp. 12-24, 2017.

5. Ashrafyan Yu.A., Harutyunyan T.N., *Isospectral Dirac operators.* Electronic Journal of Qualitative Theory of Differential Equations, No. 4, pp. 1-9, 2017.

6. Ashrafyan Yu.A., Harutyunyan T.N., *Dirac operator with linear potential and its perturbations.* Mathematical Inverse Problems, Vol. 3, No. 1, pp. 12-25, 2016.

7. Ashrafyan Yu.A., Harutyunyan T.N., *Inverse Sturm-Liouville problems with fixed boundary conditions.* Electronic Journal of Differential Equations, Vol. 2015, No. 27, pp. 1-8, 2015.

8. Ashrafyan Yu.A., Harutyunyan T.N., *On a property of norming constants of Sturm-Liouville problem.* Proceedings of Yerevan State University, No. 3, pp. 3-7, 2014.

Conference Theses

9. Ashrafyan Yu.A., Harutyunyan T.N., *Dirac operator with linear potential.* Annual session 2016 of the Armenian Mathematical Union (AMU), dedicated

to 110th anniversary of Artashes Shahinyan, Yerevan, Armenia, May 30- July 1, p. 14, 2016.

10. Ashrafyan Yu.A., *On some properties of the spectral data of Sturm-Liouville boundary-value problems.* Annual session 2015 of the AMU, dedicated to the 100th anniversary of Professor Haik Badalyan. Yerevan, Armenia, June 23-25, p. 14-15, 2015.

11. Ashrafyan Yu.A., *On a property of norming constants.* Inverse Problems: from Theory to Application. Inverse Problems, Institute of Physics, Bristol, United Kingdom, August 26-28, 2014.

12. Ashrafyan Yu.A., *Inverse Sturm-Liouville problems.* 8th International Summer School on Geometry, Mechanics and Control. Miraflores de la Sierra, Spain, June 30- Jul 04, 2014.

13. Harutyunyan T.N., Ashrafyan Yu.A., *A new proof of the Sturm oscillation theorem.* IV annual conference of the Georgian Mathematical Union dedicated to academician V. Kupradze on his 110-th birthday anniversary, Tbilisi, Batumi, Georgia, September 9-15, p. 98, 2013.

14. Harutyunyan T.N., Ashrafyan Yu.A., *On the "movement" of the zeros of eigen- functions of the Sturm-Liouville problem.* Second International Conference "Mathematics in Armenia: Advances and Perspectives", Tsaghkadzor, Armenia, August 24-31, p. 70, 2013.

15. Harutyunyan T.N., Ashrafyan Yu.A., *Inverse spectral problem for a string equation with partial information on the density function.* Annual session 2013 of the AMU, dedicated to 90th anniversary of Rafael Alexandrian, Yerevan, Armenia, May 31- July 2, p. 59, 2013.

Ամփոփում

Ատենախոսությունը նվիրված է Շտուրմ-Լիուվիլի և Դիրակի օպերատորների համար որոշ ուղիղ և հակադարձ խնդիրներին:

Առաջին գլխում ուսումնասիրվել է Շտուրմ-Լիուվիլի եզրային խնդիրը հատվածի վրա, և ստացվել են հետևյալ արդյունքները.

- Գտնվել են $\{\lambda_n^2\}_{n=0}^{\infty}$ և $\{a_n\}_{n=0}^{\infty}$ հաջորդականությունների համար անհրաժեշտ և բավարար պայմաններ, որպեսզի նրանք հանդիսանան իրականարժեք, հանրագումարելի q պոտենցիալով և նախապես ֆիքսված անջատվող եզրային պայմաններով Շտուրմ-Լիուվիլի խնդրի, համապատասխանաբար, սեփական արժեքների և նորմավորող հաստատունների բազմություն:
- Գտնվել են նորմավորող հաստատունների բազմության և α ու β եզրային պարամետրերի միջև կապը:
- Ստացվել է միակության թեորեմ $\mu_0(q, \alpha, \beta)$ փոքրագույն սեփական արժեքով կամայական ինքնահամալուծ եզրային պայմաններով Շտուրմ-Լիուվիլի հակադարձ խնդիրների համար:
- Գտնվել են $\mu_0(q, \alpha, \beta)$ փոքրագույն սեփական արժեքի գնահատականներ:
- Ցույց է տրվել, որ գոյություն ունի եզրային պայմանների բազմություն, որի համար տեղի ունի Համբարձումյանի թեորեմի ընդհանրացումը:
- Ապացուցվել է նոր տիպի միակության թեորեմ Շտուրմ-Լիուվիլի հակադարձ խնդիրներում պայմանավորված անհավասարությունով, որն ինչ-որ իմաստով հանդիսանում է Մարչենկոյի թեորեմի ընդհանրացում: Եվ ապացուցվել են այլ միակության թեորեմներ պայմանավորված անհավասարություններով:

Երկրորդ գլխում ուսումնասիրվել են Դիրակի կանոնական համակարգի համար եզրային խնդիրները հատվածի, կիսաառանցքի և առանցքի վրա, որի ընթացքում ստացվել են հետևյալ արդյունքները.

- Տրվել է հատվածի վրա որոշված Դիրակի բոլոր իզոսպեկտրալ օպերատորների նկարագրությունը բացահայտ տեսքով:
- Գտնվել են առանցքի և կիսաառանցքի վրա որոշված գծային պոտենցիալով Դիրակի օպերատորների սեփական արժեքների և սեփական ֆունկցիաների բացահայտ տեսքերը:

- Կառուցվել են գծային պոտենցիալով Դիրակի օպերատորների զրգռումները, երբ փոխում ենք վերջավոր քանակով սեփական արժեքներ և/կամ փոխում ենք նորմավորող հաստատունների արժեքները:

- Տրվել է հատվածի և կիսաառանցքի վրա Դիրակի սեփական արժեքների գրադիենտի հասկացությունը և գտնվել բանաձևեր այդ գրադիենտի համար:

- Ներմուծվել է սեփական արժեքի ածանցյալի հասկացությունը ըստ կանոնական մատրից պոտենցիալի և ցույց տրվել նրա կիրառությունը երբ նկարագրում ենք իզոսպեկտրալ Դիրակի օպերատորները կամ երբ փոխում ենք սպեկտրալ տվյալների բազմությունը:

Заключение

В диссертации рассмотрены некоторые прямые и обратные задачи для операторов Штурма-Лиувилля и Дирака.

В первой главе изучена краевая задача для Штурма-Лиувилля на конечном интервале и получены следующие результаты.

- Найдены необходимые и достаточные условия для того чтобы множества $\{\lambda_n^2\}_{n=0}^{\infty}$ и $\{a_n\}_{n=0}^{\infty}$ являлись, соответсвенно, множеством собственных значений и нормировочных постоянных задачи Штурма-Лиувилля с действительным потенциалом q из $L^1_{\mathbb{R}}[0,\pi]$ и с изначально фиксированными, разделенными краевыми условиями.

- Найдена связь между множеством нормировочных постоянных и краевых параметров α и β из краевых условий.

- Получена теорема единственности с наименьшим собственным значением $\mu_0(q,\alpha,\beta)$ для обратных задач Штурма-Лиувилля с произвольными самосопряженными краевыми условиями.

- Найдены оценки для наименьшего собственного значения $\mu_0(q,\alpha,\beta)$.

- Показано, что существует множество краевых условий, для которых имеет место обобщение теоремы Амбарцумяна.

- Доказана теорема единственности нового типа для обратных задач Штурма-Лиувилля (связаная с неравенствами), которая, в каком-то смысле, есть обобщение теоремы Марченко. И получены другие теоремы единственности, связанные с неравенствами.

Во второй главе изучены краевые задачи для канонической системы Дирака на конечном интервале, на полуоси и на всей оси, и получены следующие результаты.

- Дано описание всех изоспектральных операторов Дирака на конечном интервале в явном виде.

- В явном виде найдены собственные значения и собственные функции операторов Дирака с линейным потенциалом на оси и на полуоси.

- В явном виде построены такие возмущения операторов Дирака с линейным потенциалом, при которых меняется конечное число собственных значений и/или меняем значения нормированных постоянных.

- Дано понятие градиента собственных значений для операторов Дирака на конечном интервале и на полуоси, и найдены формулы для этих градиентов.

- Введено понятие производной собственного значения по матрице-функции и показано его применение при описании изоспектральных операторов Дирака или, при изменении конечного числа спектральных данных.