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On estimates for maximal operators associated with tangential regions

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Contents

Introduction

The following remarkable theorems of Fatou [9] play significant role in the study of boundary value problems of analytic and harmonic functions.

Theorem A (Fatou, 1906). *Any bounded analytic function on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ has non-tangential limit for almost all boundary points.*

Theorem B (Fatou, 1906). *If a function μ of bounded variation is differentiable at $x_0 \in \mathbb{T}$, then the Poisson integral*

$$\mathcal{P}_r(x, d\mu) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1 - r^2}{1 - 2r\cos(x - t) + r^2} d\mu(t)$$

converges non-tangentially to $\mu'(x_0)$ as $r \to 1$.

These two fundamental theorems, have many applications in different mathematical theories including analytic functions, Hardy spaces, harmonic analysis, differential equations and etc. There are various generalization of these theorems in different aspects. Almost everywhere convergence over some semi-tangential regions investigated by Nagel and Stein [28], Di Biase [7], Di Biase-Stokolos-Svensson-Weiss [8]. Sjögren [36, 37, 38], Rönning [30, 31, 32], Katkovskaya-Krotov [20, 24], Krotov [22, 23], Brundin [5], Mizuta-Shimomura [27], Aikawa [3] studied fractional Poisson integrals with respect to the fractional power of the Poisson kernel and obtained some tangential convergence properties for such integrals. More precisely they considered the integrals

$$\mathcal{P}_r^{(1/2)}(x, f) = \int_{\mathbb{T}} P_r^{(1/2)}(x - t) f(t)\, dt = \frac{1}{c(r)} \int_{\mathbb{T}} [P_r(x - t)]^{1/2} f(t)\, dt,$$

where

$$P_r(x) = \frac{1 - r^2}{1 - 2r\cos x + r^2}, \quad 0 < r < 1, \quad x \in \mathbb{T}$$

is the Poisson kernel for the unit disk and

$$c(r) = \int_{\mathbb{T}} [P_r(t)]^{1/2}\, dt \asymp (1-r)^{1/2} \log \frac{1}{1-r}$$

is the normalizing coefficient. Here, the notation $A \asymp B$ means double inequality $c_1 A \le B \le c_2 A$ for some positive absolute constants c_1 and c_2, which might differ in each case.

Theorem C (see [36, 30, 31]). *For any $f \in L^p(\mathbb{T})$, $1 \le p \le \infty$*

$$\lim_{r\to 1} \mathcal{P}_r^{(1/2)}(x+\theta(r), f) = f(x) \tag{0.1}$$

almost everywhere $x \in \mathbb{T}$, whenever

$$|\theta(r)| \le \begin{cases} c(1-r)\left(\log \frac{1}{1-r}\right)^p & \text{if } \ 1 \le p < \infty, \\ c_\alpha (1-r)^\alpha, \text{ for any } 0<\alpha<1 & \text{if } \ p=\infty, \end{cases} \tag{0.2}$$

where $c_\alpha > 0$ is a constant, depended only on α.

The case of $p = 1$ is proved in [36], $1 < p \le \infty$ is considered in [30], [31]. Moreover, in [30] weak type inequalities for the maximal operator of square root Poisson integrals are established.

Theorem D (Rönning, 1997). *Let $1 < p < \infty$. Then the maximal operator*

$$\mathcal{P}^*_{1/2}(x,f) = \sup_{\substack{|\theta|<c(1-r)\left(\log\frac{1}{1-r}\right)^p \\ 1/2<r<1}} \mathcal{P}_r^{(1/2)}(x+\theta, |f|)$$

is of weak type (p,p).

In [20] weighted strong type inequalities for the same operators are established. Related questions were considered also in higher dimensions. Saeki [33] studied Fatou type theorems for non-radial kernels. Korani [21] extended Fatou's theorem for the Poisson-Szegö integral. In [28] Nagel and Stein proved that the Poisson integral on the upper half space of $\mathbb{R}^{n+1}$ has the boundary limit at almost every point within a certain approach region, which is not contained in any non-tangential approach regions. Sueiro [41] extended Nagel-Stein's result for the Poisson-Szegö integral. Almost everywhere convergence over tangential tress (family of curves) were investigated by Di Biase [7], Di Biase-Stokolos-Svensson-Weiss [8]. In [20] and [3] higher dimensional cases of fractional Poisson integrals are studied as well.

In Chapter 1 we thoroughly investigate the connection between approximate identities and convergence regions. In particular, how the non-tangential convergence is connected to Poisson kernel and bounds (0.2) to the square root Poisson kernel.

We introduce $\lambda(r)-$convergence, which is a generalization of non-tangential convergence in the unit disc, where $\lambda(r)$ is a function

$$\lambda : (0,1) \to (0,\infty) \quad \text{with} \quad \lim_{r\to 1} \lambda(r) = 0. \tag{0.3}$$

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the one dimensional torus. For a given $x \in \mathbb{T}$ we define $\lambda(r,x)$ to be the interval $[x-\lambda(r), x+\lambda(r)]$. If $\lambda(r) \geq \pi$ we assume that $\lambda(r,x) = \mathbb{T}$. Let $F_r(x)$ be a family of functions from $L^1(\mathbb{T})$, where r varies in $(0,1)$. We say $F_r(x)$ is $\lambda(r)-$convergent at a point $x \in \mathbb{T}$ to a value A, if

$$\lim_{r\to 1} \sup_{\theta\in\lambda(r,x)} |F_r(\theta) - A| = 0.$$

Otherwise this relation will be denoted by

$$\lim_{\substack{r\to 1\\ \theta\in\lambda(r,x)}} F_r(\theta) = A. \tag{0.4}$$

We say $F_r(x)$ is $\lambda(r)-$divergent at $x \in \mathbb{T}$ if (0.4) does not hold for any $A \in \mathbb{R}$.

There are at least two ways to interpret $\lambda(r)-$convergence. First, we can associate the function $\lambda(r)$ with regions

$$\Omega_\lambda^x = \{re^{i\theta} \in \mathbb{C} \colon r \in (0,1),\, |\theta - x| < \lambda(r)\} \subset D, \quad x \in \mathbb{T}.$$

Then $\lambda(r)-$convergence for $F_r(x)$ at some point $x \in \mathbb{T}$ becomes convergence over the region Ω_λ^x for $\tilde{F}(re^x) = F_r(x)$. It is clear, that the non-tangential convergence in the unit disc is the case of $\lambda(r) = c(1-r)$. Second, we can think of it as one dimensional "pointwise-uniform" convergence on $\mathbb{T}$, meaning that $\lambda(r)-$convergence at a point $x \in \mathbb{T}$ depends only on values of functions on $\lambda(r, x)$ which contracts to x.

Denote by $\mathrm{BV}\,(\mathbb{T})$ the functions of bounded variation on $\mathbb{T}$. Any given function of bounded variation $\mu \in \mathrm{BV}\,(\mathbb{T})$ defines a Borel measure on $\mathbb{T}$. We consider the family of integrals

$$\Phi_r(x, d\mu) = \int_{\mathbb{T}} \varphi_r(x-t)\, d\mu(t), \quad \mu \in \mathrm{BV}\,(\mathbb{T}), \tag{0.5}$$

where $0 < r < 1$ and kernels $\varphi_r \in L^\infty(\mathbb{T})$ form an *approximate identity*, that is

$\Phi 1$. $\int_{\mathbb{T}} \varphi_r(t)\, dt \to 1$ as $r \to 1$,

$\Phi 2$. $\varphi_r^*(x) = \sup\limits_{|x| \le |t| \le \pi} |\varphi_r(t)| \to 0$ as $r \to 1, \quad 0 < |x| \le \pi$,

$\Phi 3$. $C_\varphi = \sup\limits_{0<r<1} \|\varphi_r^*\|_1 < \infty$.

In case of μ is absolutely continuous and $d\mu(t) = f(t)dt$ for some $f \in L^p(\mathbb{T})$, $1 \le p \le \infty$, then the integral (0.5) will be denoted as $\Phi_r(x, f)$.

Carlsson [6] obtained some weak type inequalities for non-negative approximate identities:

Theorem E (Carlsson, 2008). *Let $\{\varphi_r(x) \ge 0\}$ be an approximate identity and $\rho(r) = \|\varphi_r\|_q^{-p}$, where $1 \le p < \infty$ and $q = p/(p-1)$ is the conjugate number of p. Then for any $f \in L^p(\mathbb{T})$*

$$\sup_{\substack{|\theta|<c\rho(r)\\ 0<r<1}} |\Phi_r(x+\theta, f)| \le C(M|f|^p(x))^{1/p}, \quad x \in \mathbb{T},$$

where the constant C does not depend on function f.

Here $Mf(x)$ is the Hardy-Littlewood maximal function of $f \in L^1(\mathbb{T})$ defined as

$$Mf(x) = \sup_{t>0} \frac{1}{2t} \int_{x-t}^{x+t} |f(u)|\, du \quad x \in \mathbb{T}.$$

It is well known that the maximal operator M is of weak type $(1,1)$ and stong type (p,p) for $1 < p \le \infty$.

Although Theorem E gives a general connection, we will see that the regions associated with function $\rho(r)$ are not optimal in general and can be improved. The central question of Chapter 1 is the following:

Question. *For a given approximate identity $\{\varphi_r\}$ what is the necessary and sufficiant condition on $\lambda(r)$ for which*

- $\lim\limits_{\substack{r\to 1\\ y\in\lambda(r,x)}} \Phi_r(x, d\mu) = \mu'(x)$ *almost everywhere for any* $\mu \in \mathrm{BV}(\mathbb{T})$?
- $\lim\limits_{\substack{r\to 1\\ y\in\lambda(r,x)}} \Phi_r(x, f) = f(x)$ *almost everywhere for any* $f \in L^p(\mathbb{T})$, $1 \le p \le \infty$?

An analogous question can also be formulated for $f \in C(\mathbb{T})$. However, in this case Lemma 1.7 shows that (0.3) already sufficient for everywhere $\lambda(r)-$convergence.

In Section 1.3 we prove that the condition

$$\Pi(\lambda, \varphi) = \limsup_{r\to 1} \lambda(r)\|\varphi_r\|_\infty < \infty$$

is necessary and sufficient for almost everywhere $\lambda(r)-$convergence of the integrals $\Phi_r(x, d\mu)$, $\mu \in \mathrm{BV}(\mathbb{T})$ as well as $\Phi_r(x, f)$, $f \in L^1(\mathbb{T})$. Moreover, we prove that convergence holds at any point where μ is differentiable for the integrals $\Phi_r(x, d\mu)$ and at any Lebesgue point of $f \in L^1(\mathbb{T})$ for the integrals $\Phi_r(x, f)$.

Definition 1.1. *We say that a given approximate identity $\{\varphi_r\}$ is regular if each $\varphi_r(x)$ is non-negative, decreasing on $[0, \pi]$ and increasing on $[-\pi, 0]$.*

Clearly, in this case the property *Φ3* is unnecessary, since it immediately follows from *Φ1*.

Theorem 1.1 (see [19])**.** *Let $\{\varphi_r\}$ be a regular approximate identity and $\lambda(r)$ satisfies the condition $\Pi(\lambda, \varphi) < \infty$. If $\mu \in \mathrm{BV}(\mathbb{T})$ is differentiable at x_0, then*

$$\lim_{\substack{r\to 1\\ x\in\lambda(r,x_0)}} \Phi_r(x, d\mu) = \mu'(x_0).$$

An analogous theorem holds as well in the non-regular case of kernels, but at this time the points where (0.5) converges satisfy strong differentiability condition.

Definition 1.2. *We say a given function of bounded variation μ is strong differentiable at $x_0 \in \mathbb{T}$, if there exist a number c such that the variation of the function $\mu(x) - cx$ has zero derivative at $x = x_0$.*

If μ is absolutely continuous and $d\mu(t) = f(t)dt$ then this property means that x_0 is a Lebesgue point for $f(x)$, i.e.

$$\lim_{h\to 0}\frac{1}{2h}\int_{-h}^{h}|f(x)-f(x_0)|dx = 0.$$

It is well-known that strong differentiability at x_0 implies the existence of $\mu'(x_0)$, and any function of bounded variation is strong differentiable almost everywhere.

Theorem 1.2 (see [19])**.** *Let $\{\varphi_r\}$ be an arbitrary approximate identity and $\lambda(r)$ satisfies the condition $\Pi(\lambda,\varphi) < \infty$. If $\mu \in \mathrm{BV}\,(\mathbb{T})$ is strong differentiable at $x_0 \in \mathbb{T}$, then*

$$\lim_{\substack{r\to 1\\ x\in\lambda(r,x_0)}} \Phi_r\,(x, d\mu) = \mu'(x_0).$$

The following theorem implies the sharpness of the condition $\Pi(\lambda,\varphi) < \infty$ in Theorem 1.1 and Theorem 1.2.

Theorem 1.3 (see [19])**.** *If $\{\varphi_r\}$ is an arbitrary approximate identity and the function $\lambda(r)$ satisfies the condition $\Pi(\lambda,\varphi) = \infty$, then there exist a function $f \in L^1(\mathbb{T})$ such that*

$$\limsup_{\substack{r\to 1\\ y\in\lambda(r,x)}} \Phi_r\,(y, f) = \infty$$

for all $x \in \mathbb{T}$.

Thus, the condition $\Pi(\lambda,\varphi) < \infty$ determines the exact rate of $\lambda(r)$ function, ensuring such convergence. It is interesting, that this rate depends only on the values $\|\varphi_r\|_\infty$. Notice that, if the kernel φ_r coincides with the Poisson kernel P_r (which is a regular approximate identity), then $\|P_r\|_\infty \asymp \frac{1}{1-r}$ and the bound $\Pi(\lambda, P) < \infty$ coincides with the well-known condition

$$\limsup_{r\to 1} \frac{\lambda(r)}{1-r} < \infty, \tag{0.6}$$

guaranteeing non-tangential convergence in the unit disk. So, Theorem 1.1 implies and generalizes Fatou's theorem. Furthermore, if we take the fractional Poisson kernel $P_r^{(1/2)}$ (which is regular as well), then

$$\|P_r^{(1/2)}\|_\infty = \frac{1}{c(r)}\|P_r^{1/2}\|_\infty \asymp \left((1-r)\log\frac{1}{1-r}\right)^{-1}$$

and from Theorem 1.1 we deduce (0.1) when $p = 1$ with an additional information about the points where the convergence occurs.

Additionally, some weak type inequalities are established for the associated maximal operator Φ_λ^*, which is defined as

$$\Phi_\lambda^*(x, f) = \sup_{\substack{|x-y|<\lambda(r)\\ 0<r<1}} |\Phi_r(y, f)| = \sup_{\substack{|x-y|<\lambda(r)\\ 0<r<1}} \left|\int_{\mathbb{T}} \varphi_r(y-t) f(t)\, dt\right|. \tag{0.7}$$

Theorem 1.4. *Let $\{\varphi_r\}$ be an arbitrary approximate identity and for some $1 \le p < \infty$ the function $\lambda(r)$ satisfies*

$$\tilde{\Pi}_p(\lambda, \varphi) = \sup_{0<r<1} \lambda(r)\|\varphi_r\|_\infty \varphi_*(r)^{p-1} < \infty,$$

where

$$\varphi_*(r) = \sup_{x\in\mathbb{T}} |x\varphi_r^*(x)|.$$

Then for any $f \in L^1(\mathbb{T})$

$$\Phi_\lambda^*(x, f) \le C\left(M|f|^p(x)\right)^{1/p}, \quad x \in \mathbb{T},$$

where the constant C does not depend on function f. In particular, the operator Φ_λ^* is of weak type (p, p), i.e.

$$|\{x \in \mathbb{T} \colon \Phi_\lambda^*(x, f) > t\}| \le \frac{\tilde{C}}{t^p}\|f\|_p^p$$

holds for any $t > 0$, where constant $\tilde{C}$ does not depend on function f and t.

Using the standard methods, it can be shown that these weak type inequalities imply almost everywhere $\lambda(r)-$convergence with the condition

$$\Pi_p(\lambda,\varphi)=\limsup_{r\to 1}\lambda(r)\|\varphi_r\|_\infty\varphi_*^{p-1}(r)<\infty.$$

As we will see in Lemma 1.8, the function $\varphi_*(r)$ satisfies

$$\frac{c}{\log\|\varphi_r\|_\infty}\le\varphi_*(r)\le C_\varphi,\quad r_0<r<1, \tag{0.8}$$

where c is a positive absolute constant. Note that both bounds in (0.8) are accessible. For instance, if we take the Poisson kernel $P_r(t)$ then it can be checked that $P_*(r)\asymp 1$. On the other hand, if we take the square root Poisson kernel $P_r^{(1/2)}(t)$, then one can show that

$$P_*^{(1/2)}(r)\asymp\left(\log\frac{1}{1-r}\right)^{-1}\asymp\frac{1}{\log\|P_r^{(1/2)}\|_\infty}. \tag{0.9}$$

From the first inequality of (0.8) it follows that for any $1\le p<\infty$ the condition $\Pi_p(\lambda,\varphi)<\infty$ on $\lambda(r)$ cannot be weaker than

$$\limsup_{r\to 1}\lambda(r)\|\varphi_r\|_\infty\left(\frac{1}{\log\|\varphi_r\|_\infty}\right)^{p-1}<\infty.$$

The second inequality of (0.8) ensures that the multiplier $\varphi_*(r)$ in condition $\Pi_p(\lambda,\varphi)<\infty$ can only weaken that condition (in other words can only enlarge the associated region of convergence in the unit disk) if we increase p, i.e. condition $\Pi_{p_1}<\infty$ imples $\Pi_{p_2}<\infty$, whenever $1\le p_1\le p_2<\infty$.

Taking into account (0.9), note that these results imply (0.1) when $1<p<\infty$ as well as Theorem D. Moreover, combining Lemma 1.3 and Lemma 1.6, we get that Theorem 1.4 holds if we replace the condition $\tilde{\Pi}_p(\lambda,\varphi)<\infty$ by

$$\sup_{0<r<1}\lambda(r)\|\varphi_r\|_q^p<\infty, \tag{0.10}$$

where $q=p/(p-1)$ is the conjugate number of p. Thus, we obtain Theorem E for general approximate identities, not necessarily non-negative.

In Section 1.4 an analogous necessary and sufficient condition will be established also for almost everywhere $\lambda(r)-$convergence of $\Phi_r(x,f)$, $f\in L^\infty(\mathbb{T})$, and this condition looks like

$$\Pi_\infty(\lambda,\varphi)=\limsup_{\delta\to0}\limsup_{r\to1}\int_{-\delta\lambda(r)}^{\delta\lambda(r)}\varphi_r(t)dt=0,$$

which contains more information about $\{\varphi_r\}$ than $\Pi(\lambda,\varphi)$ does.

Theorem 1.5 (see [19])**.** *If $\{\varphi_r\}$ is a regular approximate identity consisting of even functions and the function $\lambda(r)$ satisfies $\Pi_\infty(\lambda,\varphi)=0$, then for any $f\in L^\infty(\mathbb{T})$ the relation*

$$\lim_{\substack{r\to1\\ y\in\lambda(r,x)}}\Phi_r\left(y,f\right)=f(x)$$

holds at any Lebesgue point $x\in\mathbb{T}$.

Theorem 1.6 (see [19])**.** *If $\{\varphi_r\}$ is a regular approximate identity consisting of even functions and the function $\lambda(r)$ satisfies $\Pi_\infty(\lambda,\varphi)>0$, then there exists a set $E\subset\mathbb{T}$, such that $\Phi_r\left(x,\mathbb{I}_E\right)$ is $\lambda(r)-$divergent at any $x\in\mathbb{T}$.*

One can easily check that in the case of Poisson kernel $P_r(t)$, for a given function $\lambda(r)$ with (0.3), the value of $\Pi_\infty(\lambda,P)$ can be either 0 or 1. Besides, the condition $\Pi_\infty(\lambda,P)=0$ is equivalent to (0.6), and $\Pi_\infty(\lambda,P)=1$ coincides with

$$\limsup_{r\to1}\frac{\lambda(r)}{1-r}=\infty.$$

Now suppose that $\lambda(r)$ satisfies the condition (0.2) with $p=\infty$. Simple calculations show that for such $\lambda(r)$ and for the square root Poisson kernel $P_r^{(1/2)}(t)$ we have $\Pi_\infty(\lambda,P^{(1/2)})=0$. Hence Theorem 1.5 implies (0.1) when $p=\infty$ with an additional information about the points where the convergence occurs. Taking $\lambda(r)=(1-r)^\alpha$ with a fixed $0<\alpha<1$ we will get $\Pi_\infty(\lambda,P^{(1/2)})=1-\alpha>0$, and applying Theorem 1.6 we conclude the optimality of the bound (0.2) in the case $p=\infty$ too.

In the definition of $\lambda(r)-$convergence the range of the parameter r is $(0,1)$ with the limit point 1, that is, we consider the convergence or divergence properties when $r\to1$. We do this way in order to compare our results with the boundary properties of analytic and harmonic functions in the unit disc. Certainly it is not essential in the theorems. We could take any set $Q \subset \mathbb{R}$ with limit point r_0 which is either a finite number or ∞. We may define an approximate identity on the real line to be a family of functions $\varphi_r \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, $r > 0$, which satisfies the same conditions *Φ1 − Φ3* as approximate identity on $\mathbb{T}$ does. We just need to make a little change in the condition *Φ2*, that is to add $\left\|\varphi_r^* \cdot \mathbb{I}_{\{|t|\geq\delta\}}\right\|_1 \to 0$ as $r \to 0$ for any $\delta > 0$. In this case usually convergence is considered while $r \to 0$. Analogously, all the results Theorem 1.1−Theorem 1.6 can be formulated and proved for the integrals

$$\Phi_r(x, d\mu) = \int_{\mathbb{R}} \varphi_r(x - t)\, d\mu(t), \quad \mu \in \mathrm{BV}(\mathbb{R}), \quad r > 0, \tag{0.11}$$

and they can be done just repeating the proofs with miserable changes.

Any function $\Phi \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ with $\|\Phi\|_1 = 1$ and $\Phi^* \in L^1(\mathbb{R})$ defines an approximate identity by

$$\varphi_r(x) = \frac{1}{r}\Phi\left(\frac{x}{r}\right) \quad \text{as} \quad r \to 0.$$

Operators corresponding to such kernels in higher dimensional case were investigated by Stain ([39], p. 57). Note for such kernels we have

$$\|\varphi_r\|_\infty = \frac{1}{r}\|\Phi\|_\infty, \quad \varphi_*(r) = \sup_{x\in\mathbb{R}} |x\Phi^*(x)| \leq \|\Phi^*\|_1$$

and therefore, for $1 \leq p < \infty$, the condition $\Pi_p(\lambda, \varphi) < \infty$ takes the form $\lambda(r) \leq c \cdot r$. The case $p = \infty$ can be done in the same way as we did it for the Poisson kernel. The value $\Pi_\infty(\lambda, \Phi)$ can be either 0 or 1, the condition $\Pi_\infty(\lambda, \Phi) = 0$ is equivalent to $\lambda(r) \leq c \cdot r$ and the condition $\Pi_\infty(\lambda, \Phi) = 1$ is equivalent to $\limsup_{r\to 0} \lambda(r)/r = \infty$. The bound $\lambda(r) \leq c \cdot r$ characterizes the non-tangential convergence in the upper half plane and it turns out to be a necessary and sufficient condition for almost everywhere $\lambda(r)$−convergence of the integrals (0.11).

In addition, we would like to bring one consequence of our results, that we consider interesting.

Corollary 1.1. *If $\sigma_n(x, f)$ are the Fejer means of Fourier series of a function $f \in L^1(\mathbb{T})$ and $\theta_n = O(1/n)$, then $\sigma_n(x+\theta_n, f) \to f(x)$ at any Lebesgue point $x \in \mathbb{T}$.*

Littlewood [25] made an important complement to the theorem of Fatou, proving essentiality of non-tangential approach in that theorem. The following formulation of Littlewood's theorem fits to the further aim of the thesis.

Theorem F (Littlewood, 1927)**.** *If a continuous function $\lambda : [0, 1] \to \mathbb{R}$ satisfies the conditions*

$$\lambda(1) = 0, \quad \lim_{r\to1} \frac{\lambda(r)}{1-r} = \infty, \tag{0.12}$$

then there exists a bounded analytic function $f(z)$, $z \in D$, such that the boundary limit

$$\lim_{r\to1} f\left(re^{i(x+\lambda(r))}\right)$$

does not exist almost everywhere on $\mathbb{T}$.

There are various generalization of these theorems in different aspects. A simple proof of this theorem was given by Zygmund [45]. In [26] Lohwater and Piranian proved, that in Littlewood's theorem almost everywhere divergence can be replaced to everywhere and the example function can be a Blaschke product. That is

Theorem G (Lohwater and Piranian, 1957)**.** *If $\lambda(r)$ is a continuous function with (0.12), then there exists a Blaschke product $B(z)$ such that the limit*

$$\lim_{r\to1} B\left(re^{i(x+\lambda(r))}\right)$$

does not exist for any $x \in \mathbb{T}$.

In [1] Aikawa obtained a similar everywhere divergence theorem for bounded harmonic functions on the unit disk, giving a positive answer to a problem raised by Barth [[4], p. 551].

Theorem H (Aikawa, 1990). *If $\lambda(r)$ is a continuous function with (0.12), then there exists a bounded harmonic function $u(z)$ on the unit disc, such that the limit*

$$\lim_{r\to 1} u\left(re^{i(x+\lambda(r))}\right)$$

does not exist for any $x \in \mathbb{T}$.

As it is noticed in [1] this theorem implies Theorem F. Indeed, if $u(z)$ is an example of harmonic function obtained from Theorem H and $v(z)$ is its harmonic conjugate, then the holomorphic function $\exp(u + iv)$ holds the same divergence property as $u(z)$ does.

It is well known that these theorems can be also formulated in the terms of Poisson integral

$$\mathcal{P}_r(x, f) = \frac{1}{2\pi}\int_{\mathbb{T}} P_r(x-t)f(t)dt,$$

since any bounded analytic or harmonic function on the unit disc can be written in this form, where f is either in H^∞ or L^∞. In addition, the proofs of these theorems are based on some properties of such functions.

Related questions were considered also in higher dimensions. Littlewood type theorems for the higher dimensional Poisson integral established by Aikawa [1, 2] and for the Poisson-Szegö integral by Hakim-Sibony [12] and Hirata [14].

Notice, that Theorem 1.6 does not imply Theorem F or Theorem H. It provides everywhere divergence of

$$\Phi_r(x + \lambda_x(r), \mathbb{I}_E) \quad \text{as} \quad r \to 1,$$

where each function $\lambda_x : (0, 1) \to (0, \infty)$ satisfies the bound $|\lambda_x(r)| \le \lambda(r)$. In Theorem F and Theorem H we have stronger divergence than in Theorem 1.6, that is, each function $\lambda_x(r)$ coincides with a given function $\lambda(r)$.

In Chapter 2 we generalize Littlewood's theorem for the integrals $\Phi_r(x, f)$ with more general kernels than approximate identities. Namely, we consider the same integrals $\Phi_r(x, f)$ with a family of kernels $\{\varphi_r\}$ satisfying

Φ1. $\int_{\mathbb{T}} \varphi_r(t)\,dt \to 1 \quad \text{as} \quad r \to 1,$

Φ4. $\varphi_r(x) \geq 0, \quad x \in \mathbb{T},\ 0 < r < 1,$

Φ5. for any numbers $\gamma > 0$ and $0 < \tau < 1$ there exists such $\delta > 0$ that

$$\int_e \varphi_r(t)\,dt < \gamma, \quad 0 < r < \tau$$

for any measurable $e \subset \mathbb{T}$ with $|e| < \delta$.

Notice, that *Φ5* is an ordinary absolute continuity condition and it is much more weaker than the condition *Φ2*. For example, it is satisfied whenever

$$\sup_{0<r<\tau} \|\varphi_r\|_\infty < \infty, \quad 0 < \tau < 1.$$

We introduce another quantity

$$\Pi^*(\lambda, \varphi) = \limsup_{\delta \to 0} \liminf_{r \to 1} \int_{-\delta\lambda(r)}^{\delta\lambda(r)} \varphi_r(t)\,dt \leq \Pi_\infty(\lambda, \varphi)$$

and prove the following theorems.

Theorem 2.1 (see [18]). *Let $\{\varphi_r\}$ be a family of kernels with Φ1, Φ4, Φ5. If a function $\lambda \in C[0,1]$ satisfies the conditions $\lambda(1) = 0$ and $\Pi^*(\lambda, \varphi) > 1/2$, then there exists a measurable set $E \subset \mathbb{T}$ such that*

$$\limsup_{r \to 1} \Phi_r\left(x + \lambda(r), \mathbb{I}_E\right) - \liminf_{r \to 1} \Phi_r\left(x + \lambda(r), \mathbb{I}_E\right) \geq 2\Pi^* - 1.$$

In the case of Poisson kernel under the condition (0.12) we have $\Pi^* = 1 > 1/2$. Therefore Theorem 2.1 implies the following generalization of Theorem F and Theorem H, giving additional information about the divergence character.

Corollary 2.1. *For any function $\lambda \in C[0,1]$ satisfying (0.12), there exists a harmonic function $u(z)$, $z \in D$ on the unit disc with $0 \leq u(z) \leq 1$, such that*

$$\limsup_{r \to 1} u\left(re^{i(x+\lambda(r))}\right) = 1, \quad \liminf_{r \to 1} u\left(re^{i(x+\lambda(r))}\right) = 0,$$

at any point $x \in \mathbb{T}$.

The higher dimensional case of this corollary was considered by Hirata [14]. We construct also a Blaschke product with Littlewood type divergence condition as in Theorem 2.1, which generalizes Theorem G. In this case a stronger condition $\Pi^*(\lambda,\varphi)=1$ is required.

Theorem 2.2 (see [18])**.** *Let a family of kernels $\{\varphi_r\}$ satisfies Φ1, Φ4, Φ5 and for $\lambda\in C[0,1]$ we have $\lambda(1)=0$ and $\Pi^*(\lambda,\varphi)=1$. Then there exists a function $B\in L^\infty(\mathbb{T})$, which is the boundary function of a Blaschke product, such that the limit*

$$\lim_{r\to 1}\Phi_r\left(x+\lambda(r),B\right)$$

does not exist for any $x\in\mathbb{T}$.

Note that, as Theorem 1.1−Theorem 1.6, Theorem 2.1 can also be formulated and proved for the integrals

$$\Phi_r(x,f)=\int_{\mathbb{R}}\varphi_r(x-t)f(t)\,dt,\quad f\in L^1(\mathbb{R}),\quad 0<r<1, \tag{0.13}$$

where the kernels $\varphi_r\in L^\infty(\mathbb{R})\cap L^1(\mathbb{R})$ satisfy the conditions *Φ1*, *Φ4*, *Φ5*. Furthermore, notice that for any positive function $\Phi\in L^\infty(\mathbb{R})\cap L^1(\mathbb{R})$ with $\|\Phi\|_1=1$ the kernels

$$\varphi_r(x)=\frac{1}{1-r}\Phi\left(\frac{x}{1-r}\right),\quad x\in\mathbb{R},\quad 0<r<1 \tag{0.14}$$

satisfy the conditions *Φ4* and *Φ5*. One can check, that for the Poisson kernel and for (0.14) the following conditions are equivalent

$$\lim_{r\to 1}\frac{\lambda(r)}{1-r}=\infty \iff \Pi^*(\lambda,\varphi)=1 \iff \Pi^*(\lambda,\varphi)>0.$$

Therefore, if the kernels in (0.13) coincide with (0.14) and $\lambda(r)$ satisfies (0.12), then Theorem 2.1 formulated for the integrals (0.13) implies everywhere strong-type divergence for (0.13), which covers the one-dimensional case of a theorem obtained by Aikawa in [3].

Now we proceed to the introduction of the third chapter.

Let $\mathcal{R}^n$ be the family of half-open (or half-closed) rectangles $\prod_{i=1}^n[a_i,b_i)$ in $\mathbb{R}^n$ and $\mathcal{DR}^n$ be the family of dyadic rectangles of the form

$$\prod_{i=1}^n\left[\frac{j_i-1}{2^{m_i}},\frac{j_i}{2^{m_i}}\right),\quad j_i,m_i\in\mathbb{Z},\quad i=1,2,\ldots,n. \tag{0.15}$$

Let $\mathcal{Q}^n \subset \mathcal{R}^n$ be the family of half-open squares in $\mathbb{R}^n$ and $\mathcal{DQ}^n$ be the family of dyadic squares ($m_1 = m_2 = \cdots = m_n$). Obviously $\mathcal{DR}^n \subset \mathcal{R}^n$ and $\mathcal{DQ}^n \subset \mathcal{Q}^n$. For a set $E \subset \mathbb{R}^n$ we denote

$$\operatorname{diam}(E) = \sup_{x,y\in E} \|x - y\|.$$

Definition 3.1. *A family $\mathcal{B}$ of bounded, positively measured sets from $\mathbb{R}^n$ is said to be a differentiation basis (or simply basis), if for any point $x \in \mathbb{R}^n$ there exists a sequence of sets $E_k \in \mathcal{B}$ such that $x \in E_k$, $k = 1, 2, \ldots$ and $\operatorname{diam}(E_k) \to 0$ as $k \to \infty$.*

Let $\mathcal{B}$ be a differentiation basis and $L_{\text{loc}}(\mathbb{R}^n)$ be the space of locally integrable functions:

$$L_{\text{loc}}(\mathbb{R}^n) = \{f \colon f \in L(K) \text{ for any compact } K \subset \mathbb{R}^n\}.$$

For any function $f \in L_{\text{loc}}(\mathbb{R}^n)$ we define

$$\delta_{\mathcal{B}}(x, f) = \limsup_{\operatorname{diam}(E)\to 0,\, x\in E\in\mathcal{B}} \left| \frac{1}{|E|} \int_E f(t)dt - f(x) \right|.$$

The integral of a function $f \in L_{\text{loc}}(\mathbb{R}^n)$ is said to be differentiable at a point $x \in \mathbb{R}^n$ with respect to the basis $\mathcal{B}$, if $\delta_{\mathcal{B}}(x, f) = 0$. The integral of a function is said to be differentiable with respect to the basis $\mathcal{B}$, if it is differentiable at almost every point. Consider the following classes of functions

$$\mathcal{F}(\mathcal{B}) = \{f \in L_{\text{loc}}(\mathbb{R}^n) : \ \delta_{\mathcal{B}}(x, f) = 0 \text{ almost everywhere } \},$$

$$\mathcal{F}^+(\mathcal{B}) = \{f \in L_{\text{loc}}(\mathbb{R}^n) : \ f(x) \geq 0,\ \delta_{\mathcal{B}}(x, f) = 0 \text{ almost everywhere } \}.$$

Note that $\mathcal{F}(\mathcal{B})$ ($\mathcal{F}^+(\mathcal{B})$) is the family of (positive) functions having almost everywhere differentiable integrals with respect to the basis $\mathcal{B}$.

Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex function. Denote by $\Psi(L)(\mathbb{R}^n)$ the class of measurable functions f defined on $\mathbb{R}^n$ such that $\Psi(|f|) \in L^1(\mathbb{R}^n)$. If Φ satisfies the Δ_2-condition $\Psi(2x) \leq k\Psi(x)$, then $\Psi(L)$ turns to be an Orlicz space with the norm

$$\|f\|_\Psi = \inf \left\{ c > 0 : \int_{\mathbb{R}^n} \Psi\left(\frac{|f|}{c}\right) \leq 1 \right\}.$$

The following classical theorems determine the optimal Orlicz space, which functions have a.e. differentiable integrals with respect to the entire family of rectangles $\mathcal{R}^n$ is the space

$$L(1+\log^+ L)^{n-1}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n),$$

corresponding to the case $\Psi(t) = t(1+\log^+ t)^{n-1}$ ([10]).

Theorem I (Jessen–Marcinkiewicz–Zygmund, [15])**.**

$$L(1+\log^+ L)^{n-1}(\mathbb{R}^n) \subset \mathcal{F}(\mathcal{R}^n).$$

Theorem J (Saks, [35])**.** *If the function* Ψ *satisfies*

$$\Psi(t) = o(t\log^{n-1} t) \text{ as } t \to \infty,$$

then $\Psi(L)(\mathbb{R}^n) \not\subset \mathcal{F}(\mathcal{R}^n)$*. Moreover, there exists a positive function* $f \in \Psi(L)(\mathbb{R}^n)$ *such that* $\delta_{\mathcal{R}^n}(x, f) = \infty$ *everywhere.*

Such theorems are valid also for the basis $\mathcal{DR}^n$. The first one trivially follows from embedding

$$L(1+\log^+ L)^{n-1}(\mathbb{R}^n) \subset \mathcal{F}(\mathcal{R}^n) \subset \mathcal{F}(\mathcal{DR}^n).$$

The second can be deduced from the following

Theorem K (Zerekidze, [42] (see also [43, 44]))**.** $\mathcal{F}^+(\mathcal{DR}^n) = \mathcal{F}^+(\mathcal{R}^n)$.

Let $\Delta = \{\nu_k : k = 1, 2, \ldots\}$ be an increasing sequence of positive integers. This sequence generates rare basis $\mathcal{DR}^n_\Delta$ of dyadic rectangles of the form (0.15) with $m_i \in \Delta$, $i = 1, 2, \ldots, n$. This kind of bases first considered in the papers [40], [11], [13], [17]. Stokolos [40] proved that the analogous of Saks theorem holds for any basis $\mathcal{DR}^n_\Delta$ with an arbitrary Δ sequence. That means $L(1+\log^+ L)^{n-1}(\mathbb{R}^n)$ is again the largest Orlicz space containing in $\mathcal{F}(\mathcal{DR}^n_\Delta)$. Oniani and Zerekidze [29] characterised translation invariant as well as net type bases formed of rectangles that are equivalent to the basis of all rectangles in the class of all non-negative functions. Karagulyan [16] proved some theorems, establishing an equivalency of some convergence conditions for multiple martingale sequences, those in particular imply some results of the papers [40], [11], [13].

In spite of the largest Orlicz spaces corresponding to the bases $\mathcal{DR}^2_\Delta$ and $\mathcal{DR}^2$ coincide, they do differentiate different set of functions, depending on density of the sequence Δ. In Section 3.3 we prove that the condition

$$\gamma_\Delta = \sup_{k\in\mathbb{N}}(\nu_{k+1} - \nu_k) < \infty$$

is necessary and sufficient for the full equivalency of rare dyadic basis $\mathcal{DR}^2_\Delta$ and complete dyadic basis $\mathcal{DR}^2$.

Theorem 3.1 (see [17])**.** *If $\Delta = \{\nu_k\}$ is an increasing sequence of positive integers with $\gamma_\Delta < \infty$, then*

$$\mathcal{F}(\mathcal{DR}^2_\Delta) = \mathcal{F}(\mathcal{DR}^2).$$

Theorem 3.2 (see [17])**.** *If $\Delta = \{\nu_k\}$ is an increasing sequence of positive integers with $\gamma_\Delta = \infty$, then there exists a function $f \in \mathcal{F}(\mathcal{DR}^2_\Delta)$ such that*

$$\limsup_{\operatorname{len}(R)\to 0,\, x\in R\in\mathcal{DR}^2} \left|\frac{1}{|R|}\int_R f(t)\,dt\right| = \infty$$

for any $x \in \mathbb{R}^n$.

Definition 3.2. *A basis $\mathcal{B}$ is said to be density basis if $\mathcal{B}$ differentiates the integral of any characteristic function $\mathbb{I}_E$ of measurable set E:*

$$\delta_\mathcal{B}(x, \mathbb{I}_E) = 0 \text{ at almost every } x \in \mathbb{R}^n.$$

We will say that the basis $\mathcal{B}$ differentiates a class of functions $\mathcal{F}$, if basis $\mathcal{B}$ differentiates the integrals of all functions of $\mathcal{F}$.

Theorem L ([10], III, Theorem 1.4)**.** *If $\mathcal{B}$ is a density basis, then it differentiates L^∞*

Note that any subbasis $\mathcal{B}'$ of a density basis $\mathcal{B}$ is also density basis, since in this case $\delta_{\mathcal{B}'}(x, f) \le \delta_\mathcal{B}(x, f)$ for any $x \in \mathbb{R}^n$ and $f \in L_{loc}(\mathbb{R}^n)$.

Definition 3.3. *Let $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{B}$ be subbases. We will say that basis $\mathcal{B}_2$ is quasi-coverable by basis $\mathcal{B}_1$ (with respect to basis $\mathcal{B}$) if for any $R \in \mathcal{B}_2$ there exist $R_k \in \mathcal{B}_1$, $k = 1, 2, \dots, p$ and $R' \in \mathcal{B}$ such that*

$$R \subseteq \tilde{R} \subseteq R', \quad \tilde{R} = \bigcup_{k=1}^{p} R_k$$

$$\operatorname{diam}(R') \le c \cdot \operatorname{diam}(R), \quad |R'| \le c|R_k|, \quad k = 1, 2, \dots, p,$$

$$\sum_{k=1}^{p} |R_k| \le c|\tilde{R}|, \quad |\tilde{R}| \le c|R|,$$

where constant $c \ge 1$ depends only on bases $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}$. We will say two bases are quasi-equivalent if they are quasi-coverable with respect to each other.

In Section 3.4 we prove that quasi-equivalent subbases $\mathcal{B}_1, \mathcal{B}_2$ of density basis $\mathcal{B}$ differentiate the same class of non-negative functions. In Section 3.5 we give several corollaries from this theorem for bases formed of rectangles.

Theorem 3.3 (see [34])**.** *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be subbases of density basis $\mathcal{B}$ formed of open sets from $\mathbb{R}^n$. If the bases $\mathcal{B}_1$ and $\mathcal{B}_2$ are quasi-equivalent with respect to $\mathcal{B}$ then*

$$\mathcal{F}^+(\mathcal{B}_1) = \mathcal{F}^+(\mathcal{B}_2).$$

Main results of the thesis are published in [17, 18, 19, 34].

CHAPTER 1

Fatou type theorems

1.1 Introduction

In this chapter we generalize Fatou's theorem for the integrals with general kernels. Here we remind Fatou's theorems about non-tangential convergence of Poisson integrals and related tangential convergence results for the square root Poisson integrals as well as weak type inequalities.

Theorem A (Fatou, 1906). *Any bounded analytic function on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ has nontangential limit for almost all boundary points.*

Theorem B (Fatou, 1906). *If a function μ of bounded variation is differentiable at $x_0 \in \mathbb{T}$, then the Poisson integral*

$$\mathcal{P}_r(x, d\mu) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1 - r^2}{1 - 2r\cos(x - t) + r^2} d\mu(t)$$

converges non-tangentially to $\mu'(x_0)$ as $r \to 1$.

Theorem C (see [36, 30, 31]). *For any $f \in L^p(\mathbb{T})$, $1 \le p \le \infty$*

$$\lim_{r \to 1} \mathcal{P}_r^{(1/2)}(x + \theta(r), f) = f(x) \tag{1.1.1}$$

almost everywhere $x \in \mathbb{T}$, whenever

$$|\theta(r)| \le \begin{cases} c(1-r)\left(\log \frac{1}{1-r}\right)^p & \text{if} \quad 1 \le p < \infty, \\ c_\alpha (1-r)^\alpha, \text{ for any } 0 < \alpha < 1 & \text{if} \quad p = \infty, \end{cases} \tag{1.1.2}$$

where $c_\alpha > 0$ is a constant, depended only on α.

Theorem D (Rönning, 1997). *Let $1 < p < \infty$. Then the maximal operator corresponding to the square root Poisson kernel*

$$\mathcal{P}^*_{1/2}(x, f) = \sup_{\substack{|\theta| < c(1-r)\left(\log \frac{1}{1-r}\right)^p \\ 1/2 < r < 1}} \mathcal{P}_r^{(1/2)}(x + \theta, |f|)$$

is of weak type (p, p).

Theorem E (Carlsson, 2008). *Let $\{\varphi_r(x) \geq 0\}$ be an approximate identity and $\rho(r) = \|\varphi_r\|_q^{-p}$, where $1 \leq p < \infty$ and $q = p/(p-1)$ is the conjugate number of p. Then for any $f \in L^p(\mathbb{T})$*

$$\sup_{\substack{|\theta| < c\rho(r) \\ 0 < r < 1}} |\Phi_r(x + \theta, f)| \leq C(M|f|^p(x))^{1/p}, \quad x \in \mathbb{T},$$

where the constant C does not depend on function f.

The organization of the current chpater is as follows. In Section 1.2 we prove auxiliarry lemmas, which will be used throughout the chapter. In Section 1.3 we prove that the condition $\Pi(\lambda, \varphi) < \infty$ determines the exact convergence regions for functional spaces BV $(\mathbb{T})$ and $L^1(\mathbb{T})$.

Definition 1.1. *We say that a given approximation of identity $\{\varphi_r\}$ is regular if each $\varphi_r(x)$ is non-negative, decreasing on $[0, \pi]$ and increasing on $[-\pi, 0]$.*

Theorem 1.1 (see [19]). *Let $\{\varphi_r\}$ be a regular approximate identity and $\lambda(r)$ satisfies the condition*

$$\Pi(\lambda, \varphi) = \limsup_{r \to 1} \lambda(r) \|\varphi_r\|_\infty < \infty.$$

If $\mu \in$ BV $(\mathbb{T})$ is differentiable at x_0, then

$$\lim_{\substack{r \to 1 \\ x \in \lambda(r, x_0)}} \Phi_r(x, d\mu) = \mu'(x_0).$$

Definition 1.2. *We say a given function of bounded variation μ is strong differentiable at $x_0 \in \mathbb{T}$, if there exist a number c such that the variation of the function $\mu(x) - cx$ has zero derivative at $x = x_0$.*

Theorem 1.2 (see [19]). *Let $\{\varphi_r\}$ be an arbitrary approximate identity and $\lambda(r)$ satisfies the condition $\Pi(\lambda,\varphi)<\infty$. If $\mu\in \mathrm{BV}(\mathbb{T})$ is strong differentiable at $x_0\in\mathbb{T}$, then*

$$\lim_{\substack{r\to 1\\ x\in\lambda(r,x_0)}} \Phi_r(x,d\mu)=\mu'(x_0).$$

Theorem 1.3 (see [19]). *If $\{\varphi_r\}$ is an arbitrary approximate identity and the function $\lambda(r)$ satisfies the condition $\Pi(\lambda,\varphi)=\infty$, then there exist a function $f\in L^1(\mathbb{T})$ such that*

$$\limsup_{\substack{r\to 1\\ y\in\lambda(r,x)}} \Phi_r(y,f)=\infty \tag{1.1.3}$$

for all $x\in\mathbb{T}$.

Additionally, we prove that the bound $\tilde{\Pi}_p(\lambda,\varphi)<\infty$ provides weak type inequalities in spaces $L^p(\mathbb{T})$, $1\le p<\infty$.

Theorem 1.4. *Let $\{\varphi_r\}$ be an arbitrary approximate identity and for some $1\le p<\infty$ the function $\lambda(r)$ satisfies*

$$\tilde{\Pi}_p(\lambda,\varphi)=\sup_{0<r<1}\lambda(r)\|\varphi_r\|_\infty\varphi_*(r)^{p-1}<\infty. \tag{1.1.4}$$

Then for any $f\in L^1(\mathbb{T})$

$$\Phi^*_\lambda(x,f)\le C\left(M|f|^p(x)\right)^{1/p},\quad x\in\mathbb{T}, \tag{1.1.5}$$

where the constant C does not depend on function f. In particular, the operator Φ^*_λ is of weak type (p,p), i.e.

$$|\{x\in\mathbb{T}\colon \Phi^*_\lambda(x,f)>t\}|\le\frac{\tilde{C}}{t^p}\|f\|_p^p$$

holds for any $t>0$, where constant $\tilde{C}$ does not depend on function f and t.

In Section 1.4 we prove that the condition $\Pi_\infty(\lambda,\varphi)=0$ is necessary and sufficient for almost everywhere $\lambda(r)-$convergence of the integrals $\Phi_r(x,f)$, $f\in L^\infty(\mathbb{T})$.

Theorem 1.5 (see [19]). *If $\{\varphi_r\}$ is a regular approximate identity consisting of even functions and*

$$\Pi_\infty(\lambda,\varphi)=\limsup_{\delta\to 0}\limsup_{r\to 1}\int_{-\delta\lambda(r)}^{\delta\lambda(r)}\varphi_r(t)dt=0,$$

then for any $f \in L^\infty(\mathbb{T})$ the relation

$$\lim_{\substack{r\to 1\\ y\in\lambda(r,x)}} \Phi_r(y,f) = f(x)$$

holds at any Lebesgue point $x \in \mathbb{T}$.

Theorem 1.6 (see [19])**.** *If $\{\varphi_r\}$ is a regular approximate identity consisting of even functions and $\Pi_\infty(\lambda,\varphi) > 0$, then there exists a set $E \subset \mathbb{T}$, such that $\Phi_r(x, \mathbb{I}_E)$ is $\lambda(r)-$divergent at any $x \in \mathbb{T}$.*

1.2 Auxiliary lemmas

The following lemma plays significant role in the proofs of Theorem 1.1 and Theorem 1.2.

Lemma 1.1. *Let a positive function $\varphi \in L^\infty(\mathbb{T})$ is decreasing on $[0,\pi]$ and increasing on $[-\pi,0]$. Then for any numbers $\varepsilon \in (0,1)$ and $\theta \in (-\pi,\pi)$ there exist a finite family of intervals $I_j \subset \mathbb{T}$, $j = 1,2,\ldots,n$, containing 0 in their closures $\bar{I}_j$, and numbers $\varepsilon_j = \pm\varepsilon$ such that*

$$\begin{aligned}
&|I_j| \le 2\sup\{|t| : \varphi(t) \ge \varepsilon\}, \quad j = 1,2,\ldots,n,\\
&\sum_{j=1}^n |I_j| < 10\varepsilon^{-1}\max\{1, |\theta|\cdot\|\varphi\|_\infty, \|\varphi\|_1\},\\
&\left|\varphi(x-\theta) - \sum_{j=1}^n \varepsilon_j \mathbb{I}_{I_j}(x)\right| \le \varepsilon.
\end{aligned}$$

Proof. Denote

$$\begin{aligned}
&y_k = \sup\{t > 0 : \varphi(t) \ge \varepsilon k\},\\
&x_k = \sup\{t > 0 : \varphi(-t) \ge \varepsilon k\}, \quad k = 1,2,\ldots,l = \left[\frac{\|\varphi\|_\infty}{\varepsilon}\right].
\end{aligned}$$

Then we obviously have

$$y_0 = \pi, \quad 0 \le y_l \le y_{l-1} \le \ldots \le y_1 \le \sup\{|t| : \varphi(t) \ge \varepsilon\}, \tag{1.2.1}$$

$$x_0 = \pi, \quad 0 \le x_l \le x_{l-1} \le \ldots \le x_1 \le \sup\{|t| : \varphi(t) \ge \varepsilon\}, \tag{1.2.2}$$

$$\left|\varphi(x-\theta) - \varepsilon\sum_{k=1}^l \mathbb{I}_{(\theta-x_k,\theta+y_k)}(x)\right| \le \varepsilon. \tag{1.2.3}$$

Without loss of generality we can suppose $0 \le \theta < \pi$. Then we denote

$$k_0 = \max\{k : 0 \le k \le l,\, \theta - x_k \le 0\}.$$

We define the desired intervals I_j, $j = 1, 2, \dots, n = 2l - k_0$, by

$$I_j = \begin{cases} (\theta - x_j, \theta + y_j) & \text{if} \quad j \le k_0, \\ (0, \theta + y_j) & \text{if} \quad k_0 < j \le l, \\ (0, \theta - x_{j-l+k_0}] & \text{if} \quad l < j \le n = 2l - k_0. \end{cases}$$

Using the equality

$$\begin{aligned} \mathbb{I}_{(\theta - x_k, \theta + y_k)}(x) &= \mathbb{I}_{(0,\theta+y_k)}(x) - \mathbb{I}_{(0,\theta - x_k]}(x) \\ &= \mathbb{I}_{I_k}(x) - \mathbb{I}_{I_{k+l-k_0}}(x), \quad k_0 < k \le l, \end{aligned}$$

we get

$$\varepsilon \sum_{k=1}^{l} \mathbb{I}_{(\theta - x_k, \theta + y_k)}(x) = \sum_{j=1}^{n} \varepsilon_j \mathbb{I}_{I_j}(x), \tag{1.2.4}$$

where

$$\varepsilon_j = \begin{cases} \varepsilon & \text{if} \quad 1 \le j \le l, \\ -\varepsilon & \text{if} \quad l < j \le n. \end{cases} \tag{1.2.5}$$

We note that $\varepsilon_j = -\varepsilon$ in the case when I_j coincides with one of the intervals $(0, \theta - x_k]$, $k_0 < k \le l$. Hence we have

$$\sum_{j=l+1}^{n} |I_j| = \sum_{k=k_0+1}^{l} (\theta - x_k) \le l \cdot \theta \le \frac{\theta \|\varphi\|_\infty}{\varepsilon}. \tag{1.2.6}$$

From (1.2.3) and (1.2.4) we get

$$\left| \varphi(x - \theta) - \sum_{j=1}^{n} \varepsilon_j \mathbb{I}_{I_j}(x) \right| \le \varepsilon \tag{1.2.7}$$

and therefore by (1.2.5) we obtain

$$\left| \int_{\mathbb{T}} \varphi(t) dt - \varepsilon \sum_{j=1}^{l} |I_j| + \varepsilon \sum_{j=l+1}^{n} |I_j| \right| \le 2\pi\varepsilon < 2\pi.$$

This and (1.2.6) imply

$$\varepsilon \sum_{j=1}^{n} |I_j| \le 2\varepsilon \sum_{j=l+1}^{n} |I_j| + \|\varphi\|_1 + 2\pi \le 2\theta \|\varphi\|_\infty + \|\varphi\|_1 + 2\pi,$$

which together with (1.2.1), (1.2.2) and (1.2.7) completes the proof of lemma. □

We will use the following lemma in the proof of Theorem 1.3.

Lemma 1.2. *Let $\varphi \in \mathrm{BV}(\mathbb{T})$ be a function of bounded variation and*

$$\Delta_k = \bigcup_{j=0}^{n_k-1} \left[\frac{2\pi j}{n_k} - \delta_k, \frac{2\pi j}{n_k} + \delta_k\right] \subset \mathbb{T},$$

where $n_k \in \mathbb{N}, \delta_k \in \mathbb{T}$ such that $n_k \to \infty$ as $k \to \infty$ and $\delta_k > 0$, $k = 1, 2, \ldots$. Then

$$\lim_{k\to\infty} \frac{1}{|\Delta_k|} \int_{\Delta_k} \varphi(\theta + t)\, dt = \frac{1}{2\pi} \int_{\mathbb{T}} \varphi(t)\, dt,$$

where the convergence is uniform with respect to $\theta \in \mathbb{T}$.

Proof. Denote by Δ_k^j the jth component interval of Δ_k such that $\Delta_k = \cup_{0\le j<n_k} \Delta_k^j$ and $|\Delta_k^j| = 2\delta_k$. Let $\theta + \Delta_k^j = \{\theta + t \colon t \in \Delta_k^j\}$ and $\mathrm{V}(\varphi, [a, b])$ be the total variation of function φ on an interval $[a, b] \subset \mathbb{T}$. Then

$$\begin{aligned}
&\left|\frac{1}{|\Delta_k|} \int_{\Delta_k} \varphi(\theta + t)\, dt - \frac{1}{2\pi} \int_{\mathbb{T}} \varphi(t)\, dt\right| \\
&\le \left|\frac{1}{n_k} \sum_{j=0}^{n_k-1} \frac{1}{2\delta_k} \int_{\Delta_k^j} \varphi(\theta + t)\, dt - \frac{1}{n_k} \sum_{j=0}^{n_k-1} \varphi\left(\theta + \frac{2\pi j}{n_k}\right)\right| \\
&+ \left|\frac{1}{n_k} \sum_{j=0}^{n_k-1} \varphi\left(\theta + \frac{2\pi j}{n_k}\right) - \frac{1}{2\pi} \int_{\mathbb{T}} \varphi(\theta + t)\, dt\right| \\
&\le \frac{1}{n_k} \sum_{j=0}^{n_k-1} \frac{1}{2\delta_k} \int_{\Delta_k^j} \left|\varphi(\theta + t) - \varphi\left(\theta + \frac{2\pi j}{n_k}\right)\right| dt \qquad (1.2.8) \\
&+ \frac{1}{2\pi} \sum_{j=0}^{n_k-1} \int_{2\pi j/n_k}^{2\pi(j+1)/n_k} \left|\varphi(\theta + t) - \varphi\left(\theta + \frac{2\pi j}{n_k}\right)\right| dt \\
&\le \frac{1}{n_k} \sum_{j=0}^{n_k-1} \mathrm{V}\left(\varphi, \theta + \Delta_k^j\right) + \frac{1}{n_k} \sum_{j=0}^{n_k-1} \mathrm{V}\left(\varphi, \left[\theta + \frac{2\pi j}{n_k}, \theta + \frac{2\pi(j+1)}{n_k}\right]\right) \\
&\le \frac{2}{n_k} \mathrm{V}(\varphi, \mathbb{T}).
\end{aligned}$$

The last term does not depend on θ and vanishes as $k \to \infty$, which completes the proof of the lemma. □

The next 3 lemmas are key ingredients of the proof of Theorem 1.4.

Lemma 1.3. *Let $\{\varphi_r\}$ be an arbitrary approximate identity and $\lambda(r) > 0$ be any function. Then for any function $f \in L^1(\mathbb{T})$*

$$\sup_{\substack{|x-y|<\lambda(r)\\ 0<r<1}} \left| \int_{\lambda(r)\le|t|\le\pi} \varphi_r(t) f(y-t)\, dt \right| \le 8C_\varphi \cdot Mf(x), \quad x \in \mathbb{T}. \tag{1.2.9}$$

Proof. Without loss of generality we may assume that f is non-negative. Let $x, y \in \mathbb{T}$, $0 < r < 1$ such that $|x-y| < \lambda(r)$. We devide the interval $[\lambda(r), \pi]$ into $[2^{k-1}\lambda(r), 2^k\lambda(r)], k = 1, 2, \ldots, Q = \lceil \log \frac{\pi}{\lambda(r)} \rceil$ and estimate the values of $\varphi_r(t)$ by its maximum in each divided interval:

$$\begin{aligned}
\left| \int_{\lambda(r)}^{\pi} \varphi_r(t) f(y-t)\, dt \right| &\le \sum_{k=1}^{Q} \int_{2^{k-1}\lambda(r)}^{2^k\lambda(r)} \varphi_r^*(t) f(y-t)\, dt \\
&\le \sum_{k=1}^{Q} \varphi_r^*\left(2^{k-1}\lambda(r)\right) \int_{2^{k-1}\lambda(r)}^{2^k\lambda(r)} f(y-t)\, dt \\
&\le \sum_{k=1}^{Q} \varphi_r^*\left(2^{k-1}\lambda(r)\right) \int_{\lambda(r)}^{2^k\lambda(r)} f(y-t)\, dt
\end{aligned}$$

Since $|x-y| < \lambda(r)$ we have

$$\int_{\lambda(r)}^{2^k\lambda(r)} f(y-t)\, dt \le \int_0^{(1+2^k)\lambda(r)} f(x-t)\, dt.$$

Therefore

$$\begin{aligned}
\left| \int_{\lambda(r)}^{\pi} \varphi_r(t) f(y-t)\, dt \right| &\le \sum_{k=1}^{Q} \varphi_r^*\left(2^{k-1}\lambda(r)\right) \int_0^{(1+2^k)\lambda(r)} f(x-t)\, dt \\
&\le Mf(x) \cdot \sum_{k=1}^{Q} \varphi_r^*\left(2^{k-1}\lambda(r)\right)(1+2^k)\lambda(r) \\
&\le 8Mf(x) \cdot \sum_{k=0}^{Q-1} \varphi_r^*\left(2^k\lambda(r)\right) 2^{k-1}\lambda(r) \\
&\le 8Mf(x) \cdot \int_0^{\pi} \varphi_r^*(t)\, dt,
\end{aligned}$$

where in the last inequality we have used the following simple geometric inequlaity:

$$\begin{aligned}
&\varphi_r^*\left(\lambda(r)\right)\lambda(r) + \sum_{k=1}^{Q-1} \varphi_r^*\left(2^k\lambda(r)\right) 2^{k-1}\lambda(r) \\
&\qquad \le \int_0^{\lambda(r)} \varphi_r^*(t)\, dt + \sum_{k=1}^{Q-1} \int_{2^{k-1}\lambda(r)}^{2^k\lambda(r)} \varphi_r^*(t)\, dt \\
&\qquad \le \int_0^{\pi} \varphi_r^*(t)\, dt.
\end{aligned}$$

Thus we have

$$\left|\int_{\lambda(r)}^{\pi}\varphi_r(t)f(y-t)\,dt\right|\le 8Mf(x)\cdot\int_0^{\pi}\varphi_r^*(t)\,dt.$$

In the same way we get

$$\left|\int_{-\pi}^{-\lambda(r)}\varphi_r(t)f(y-t)\,dt\right|\le 8Mf(x)\cdot\int_{-\pi}^{0}\varphi_r^*(t)\,dt.$$

Therefore

$$\sup_{\substack{|x-y|<\lambda(r)\\0<r<1}}\left|\int_{\lambda(r)\le|t|\le\pi}\varphi_r(t)f(y-t)\,dt\right|\le 8Mf(x)\cdot\sup_{0<r<1}\|\varphi_r^*\|_1\le 8C_\varphi\cdot Mf(x).$$

□

Lemma 1.4. *Let $\{\varphi_r\}$ be an arbitrary approximate identity and $\mu(r)$, $\lambda(r)$ are some functions with*

1. $0<\mu(r)\le\lambda(r)\le\pi$,
2. $\lambda(r)\le C\mu(r)\varphi_*^{-p}(r)$, *for some $C>0$ and $p\ge 1$.*

Then for any $A\ge 1$ and for any function $f\in L^1(\mathbb{T})$

$$T_Af(x)\le\left(C\cdot\frac{M|f|^p(x)}{A}\right)^{1/p},\quad x\in\mathbb{T},\tag{1.2.10}$$

where

$$T_Af(x)=\sup_{\substack{A\mu(r)<|x-y|<\lambda(r)\\0<r<1}}\varphi_*(r)m_f(y,A\mu(r)),$$

$$m_f(y,t)=\frac{1}{2t}\int_{y-t}^{y+t}|f(u)|\,du.$$

Proof. Without loss of generality we may assume that f is non-negative. Using the definition of T_A and Jensen's inequality we get

$$\begin{aligned}T_A^pf(x)&=\sup_{\substack{A\mu(r)<|x-y|<\lambda(r)\\0<r<1}}\varphi_*^p(r)m_f^p(y,A\mu(r))\\&\le\sup_{\substack{A\mu(r)<|x-y|<\lambda(r)\\0<r<1}}\varphi_*^p(r)m_{f^p}(y,A\mu(r))\\&=\sup_{k\in\mathbb{N}}\sup_{\substack{2^{k-1}A\mu(r)<|x-y|\le 2^kA\mu(r)\\2^kA\mu(r)\le\lambda(r)\\0<r<1}}\varphi_*^p(r)m_{f^p}(y,A\mu(r))\end{aligned}$$

To estimate the inner supremum, first note that $2^k A\mu(r) \le \lambda(r) \le C\mu(r)\varphi_*(r)^{-p}$ implies $\varphi_*^p(r) \le C\left(2^k A\right)^{-1}$, where C is the constant from condition *2*. Furthermore, since $2^{k-1}A\mu(r) < |x-y| \le 2^k A\mu(r)$ we have

$$\begin{aligned} m_{f^p}(y, A\mu(r)) &= \frac{1}{2A\mu(r)} \int_{y-A\mu(r)}^{y+A\mu(r)} f^p(u)\,du \\ &\le \frac{1}{2A\mu(r)} \int_{x}^{x+(1+2^k)A\mu(r)} f^p(u)\,du \\ &\le \frac{(1+2^k)A\mu(r)}{2A\mu(r)} Mf^p(x) \le 2^k Mf^p(x). \end{aligned}$$

Therefore

$$T_A^p f(x) \le \sup_{k\in\mathbb{N}} C\left(2^k A\right)^{-1} 2^k Mf^p(x) = C \cdot \frac{Mf^p(x)}{A}.$$

□

Lemma 1.5. *Let $\{\varphi_r\}$ be an arbitrary approximate identity and $\mu(r), \lambda(r)$ are some functions satisfying the conditions 1. and 2. from Lemma 1.4. Then for any function $f \in L^1(\mathbb{T})$*

$$\sup_{\substack{|x-y|<\lambda(r)\\ 0<r<1}} \left| \int_{\mu(r)\le|t|\le\lambda(r)} \varphi_r(t) f(y-t)\,dt \right| \le \frac{4C^{1/p}}{2^{1/p}-1} \left(M|f|^p(x)\right)^{1/p}, \quad x \in \mathbb{T}. \tag{1.2.11}$$

Proof. Again, we may assume that f is non-negative. Let $x, y \in \mathbb{T}, 0 < r < 1$ and $|x-y| < \lambda(r)$. If $Q = \lceil \log \frac{\lambda(r)}{\mu(r)} \rceil$, we split the integral in (1.2.11) as follows

$$\begin{aligned} &\left| \int_{\mu(r)\le|t|\le\lambda(r)} \varphi_r(t) f(y-t)\,dt \right| \\ &\le \sum_{k=1}^{Q} \int_{2^{k-1}\mu(r)\le|t|\le 2^k\mu(r)} \varphi_r^*(t) f(y-t)\,dt \\ &\le \sum_{k=1}^{Q} \max\left(\varphi_r^*\left(2^{k-1}\mu(r)\right), \varphi_r^*\left(-2^{k-1}\mu(r)\right)\right) \int_{|t|\le 2^k\mu(r)} f(y-t)\,dt \qquad (1.2.12) \\ &= 2\sum_{k=1}^{Q} 2^{k-1}\mu(r) \max\left(\varphi_r^*\left(2^{k-1}\mu(r)\right), \varphi_r^*\left(-2^{k-1}\mu(r)\right)\right) m_f(y, 2^k\mu(r)) \\ &\le 2\sum_{k=1}^{Q} \varphi_*(r) m_f(y, 2^k\mu(r)). \end{aligned}$$

Then we split the domain of supremum in the followoing way:

$$\begin{aligned}\sup_{\substack{|x-y|<\lambda(r)\\0<r<1}} \varphi_*(r)m_f(y,A\mu(r)) \le &\sup_{\substack{|x-y|\le A\mu(r)\le\lambda(r)\\0<r<1}} \varphi_*(r)m_f(y,A\mu(r))\\ &+\sup_{\substack{A\mu(r)<|x-y|<\lambda(r)\\0<r<1}} \varphi_*(r)m_f(y,A\mu(r)).\end{aligned} \tag{1.2.13}$$

Notice that the second supremum is $T_Af(x)$. To estimate the first supremum, note that $|x-y|\le A\mu(r)\le\lambda(r)\le C\mu(r)\varphi_*(r)^{-p}$ implies

$$\varphi_*(r)\le C^{1/p}A^{-1/p}, \tag{1.2.14}$$

where C is the constant from the condition *2* of Lemma 1.4. On the other hand, from $|x-y|\le A\mu(r)$ it follows $m_f(y,A\mu(r))\le Mf(x)$, which together with (1.2.14), (1.2.13) and Lemma 1.4 gives

$$\begin{aligned}\sup_{\substack{|x-y|<\lambda(r)\\0<r<1}} \varphi_*(r)m_f(y,A\mu(r)) &\le C^{1/p}A^{-1/p}Mf(x)+T_Af(x)\\ &\le C^{1/p}A^{-1/p}(Mf^p(x))^{1/p}+\left(C\cdot\frac{Mf^p(x)}{A}\right)^{1/p}\\ &\le 2C^{1/p}A^{-1/p}(Mf^p(x))^{1/p}.\end{aligned} \tag{1.2.15}$$

Using (1.2.12) and (1.2.15) we get

$$\begin{aligned}\sup_{\substack{|x-y|<\lambda(r)\\0<r<1}} &\left|\int_{\mu(r)\le|t|\le\lambda(r)} \varphi_r(t)f(y-t)\,dt\right|\\ &\le 2\sum_{k=1}^{Q}\sup_{\substack{|x-y|<\lambda(r)\\0<r<1}} \varphi_*(r)m_f(y,2^k\mu(r))\\ &\le 4C^{1/p}\sum_{k=1}^{\infty}2^{-k/p}(Mf^p(x))^{1/p}\\ &=\frac{4C^{1/p}}{2^{1/p}-1}(Mf^p(x))^{1/p},\end{aligned}$$

which gives (1.2.11). □

Lemma 1.6. *Let $\{\varphi_r\}$ be an arbitrary approximate identity and for some $1\le p<\infty$ the function $\lambda(r)$ satisfies*

$$\sup_{0<r<1}\lambda(r)\|\varphi_r\|_q^p<\infty, \tag{1.2.16}$$

where $q = p/(p-1)$ is the conjugate number of p. Then for any function $f \in L^1(\mathbb{T})$

$$\sup_{\substack{|x-y|<\lambda(r)\\0<r<1}} \left| \int_{|t|\le\lambda(r)} \varphi_r(t) f(y-t)\, dt \right| \le C \left(M|f|^p(x) \right)^{1/p}, \quad x \in \mathbb{T}, \tag{1.2.17}$$

where C does not depend on function f.

Proof. The proof immediately follows from applying Hölder's inequality to the integral:

$$\begin{aligned}
\sup_{\substack{|x-y|<\lambda(r)\\0<r<1}} \left| \int_{|t|\le\lambda(r)} \varphi_r(t) f(y-t)\, dt \right| &\le \sup_{\substack{|x-y|<\lambda(r)\\0<r<1}} \|\varphi_r\|_q \cdot \left(\int_{|t|\le\lambda(r)} |f(y-t)|^p\, dt \right)^{1/p} \\
&\le \sup_{0<r<1} \|\varphi_r\|_q \cdot \left(\int_{|t|\le2\lambda(r)} |f(x-t)|^p\, dt \right)^{1/p} \\
&\le \sup_{0<r<1} \|\varphi_r\|_q (4\lambda(r))^{1/p} \cdot \left(M|f|^p(x) \right)^{1/p},
\end{aligned}$$

which implies (1.2.17) taking into account (1.2.16). □

Lemma 1.7. *Let $\{\varphi_r\}$ be an arbitrary approximate identity and $\lambda : (0,1) \to (0,\infty)$ be a function with $\lambda(r) \to 0$ as $r \to 0$. Then for any continuous function $f \in C(\mathbb{T})$*

$$\lim_{\substack{r\to1\\y\in\lambda(r,x)}} \Phi_r(y,f) = f(x) \tag{1.2.18}$$

for any $x \in \mathbb{T}$.

Proof. Let $f \in C(\mathbb{T})$ and $x \in \mathbb{T}$. Fix $\delta > 0$ and $\theta \in \mathbb{R}$. Then we have

$$\begin{aligned}
|\Phi_r(x+\theta,f) - f(x)| &\le \left| \int_{\mathbb{T}} \varphi_r(t) \left[f(x+\theta-t) - f(x) \right] dt \right| + o(1) \\
&\le \int_{|t|<\delta} \varphi_r^*(t) \left| f(x+\theta-t) - f(x) \right| dt \\
&+ \int_{\delta\le|t|\le\pi} \varphi_r^*(t) \left| f(x+\theta-t) - f(x) \right| dt + o(1) \\
&\le \omega(f,\delta+\theta) \|\varphi_r^*\|_1 + 2\|f\|_C \cdot 2\pi\varphi_r^*(\delta) + o(1) \\
&\le C_\varphi \cdot \omega(f,\delta+\theta) + 4\pi\|f\|_C \cdot \varphi_r^*(\delta) + o(1),
\end{aligned}$$

where $\omega(f,h)$ is the modulus of continuity in $C(\mathbb{T})$ defined as

$$\omega(f,h) = \sup_{|x-y|\le h} |f(x) - f(y)|.$$

Therefore, from $\lambda(r) \to 0$ as $r \to 1$, we conclude

$$\limsup_{\substack{r\to 1\\ |\theta|\le\lambda(r)}} |\Phi_r(x+\theta, f) - f(x)| \le \limsup_{r\to 1} \big(C_\varphi \cdot \omega(f, \delta+\lambda(r)) + 4\pi\|f\|_C \cdot \varphi_r^*(\delta)\big)$$

$$\le C_\varphi \cdot \omega(f, 2\delta).$$

Since $\omega(f,h) \to 0$ as $h \to 0$ and δ can be taken arbitrarily small, we get (1.2.18). □

Lemma 1.8. *If $\{\varphi_r\}$ is an arbitrary approximate identity, then for some $r_0 \in (0,1)$*

$$\frac{c}{\log\|\varphi_r\|_\infty} \le \varphi_*(r) \le C_\varphi, \quad r_0 < r < 1,$$

where c is a positive absolute constant.

Proof. Let $0 < r < 1$. Using the definitions of $\varphi_r^*(x)$ and $\varphi_*(r)$ we conclude

$$\varphi_r(t) \le \varphi_r^*(t) \le \frac{\varphi_*(r)}{|t|}, \quad t \in \mathbb{T}\setminus\{0\}.$$

Therefore, for a fixed $\delta > 0$ we have

$$1 + o(1) = \int_{\mathbb{T}} \varphi_r(t)\,dt \le \int_{|t|<\delta} \varphi_r^*(t)\,dt + \int_{\delta\le|t|\le\pi} \varphi_r^*(t)\,dt$$

$$\le \|\varphi_r^*\|_\infty \int_{|t|<\delta} dt + \varphi_*(r) \int_{\delta\le|t|\le\pi} \frac{dt}{|t|}$$

$$= 2\delta\|\varphi_r\|_\infty + 2\varphi_*(r)\log\frac{\pi}{\delta},$$

which implies

$$\varphi_*(r) \ge \left(\frac{1}{2} + o(1) - \delta\|\varphi_r\|_\infty\right)\left(\log\frac{\pi}{\delta}\right)^{-1}.$$

Now, if we take $\delta = \pi/\|\varphi_r\|_\infty^2$, we get

$$\varphi_*(r) \ge \left(\frac{1}{2} + o(1) - \frac{\pi}{\|\varphi_r\|_\infty}\right)\frac{1}{2\log\|\varphi_r\|_\infty},$$

which completes the proof of the first inequality (for example with $c = 1/5$), since $\|\varphi_r\|_\infty \to \infty$ as $r \to 1$. The second inequality can be deduced from the following:

$$\varphi_*(r) = \sup_{x\in\mathbb{T}} |x\varphi_r^*(x)| \le \sup_{x\in\mathbb{T}} \left|\int_{|t|\le|x|} \varphi_r^*(t)\,dt\right| \le C_\varphi.$$

□

If $f(x)$ is a function defined on a set $E\subset\mathbb{T}$ we denote

$$\mathrm{OSC}_{\,x\in E}f(x)=\sup_{x,y\in E}|f(x)-f(y)|.$$

Lemma 1.9. *Let*

$$U_n^\delta=\bigcup_{k=0}^{n-1}\left(\frac{\pi(2k+1-\delta)}{n},\frac{\pi(2k+1+\delta)}{n}\right),\quad n\in\mathbb{N},\quad 0<\delta<\frac{1}{2},$$

and $J\subset\mathbb{T}$, $\pi>|J|\ge 16\pi/n$, is an arbitrary closed interval. If a measurable set $E\subset\mathbb{T}$ satisfies either

$$E\cap J=J\cap U_n^\delta \quad or \quad E\cap J=J\setminus U_n^\delta,$$

and $\varphi\in L^\infty(\mathbb{T})$ is an even decreasing on $[0,\pi]$ function, then

$$\mathrm{OSC}_{\,\theta\in\left[x-\frac{4\pi}{n},x+\frac{4\pi}{n}\right]}\int_{\mathbb{T}}\varphi(\theta-t)\mathbb{I}_E(t)dt>\int_{-\frac{\pi\delta}{n}}^{\frac{\pi\delta}{n}}\varphi(t)dt-16\delta-2\pi\varphi\left(\frac{|J|}{4}\right),\tag{1.2.19}$$

for any $x\in J$.

Proof. We suppose $J=[a,b]$ and

$$\frac{2\pi(p-1)}{n}<a\le\frac{2\pi p}{n},\quad \frac{2\pi(q-1)}{n}<b\le\frac{2\pi q}{n}.$$

First we consider the case

$$E\cap J=J\cap U_n^\delta.\tag{1.2.20}$$

If $x\in J$, then

$$x\in I=\left[\frac{2\pi(m-1)}{n},\frac{2\pi m}{n}\right]$$

for some $p\le m\le q$. Without loss of generality we may assume that the center of I is on the left hand side of the center of J. Then we will have

$$b-\frac{2\pi(m+1)}{n}\ge\frac{|J|}{2}-\frac{4\pi}{n}\ge\frac{|J|}{4}.\tag{1.2.21}$$

It is clear, that the points

$$\theta_1=\frac{2\pi m}{n}+\frac{\pi}{n},\quad \theta_2=\frac{2\pi(m+1)}{n}\tag{1.2.22}$$

are in the interval $[0, x+4\pi/n]$. Besides we have

$$\begin{aligned}\int_{\mathbb{T}} \varphi(\theta_1 - t)\mathbb{I}_E(t)dt - \int_{\mathbb{T}} \varphi(\theta_2 - t)\mathbb{I}_E(t)dt \\ = \int_{\theta_2-\pi}^{a} [\varphi(\theta_1 - t) - \varphi(\theta_2 - t)]\mathbb{I}_E(t)dt \\ + \int_{a}^{b} [\varphi(\theta_1 - t) - \varphi(\theta_2 - t)]\mathbb{I}_E(t)dt \\ + \int_{b}^{\theta_2+\pi} [\varphi(\theta_1 - t) - \varphi(\theta_2 - t)]\mathbb{I}_E(t)dt \\ = A_1 + A_2 + A_3.\end{aligned}$$

Since φ is decreasing on $[0, \pi]$ we have

$$A_1 \geq 0. \tag{1.2.23}$$

If $t \in [b, \theta_2 + \pi]$ then, using (1.2.21), we get

$$t - \theta_2 \geq b - \theta_2 \geq \frac{|J|}{4}, \quad t - \theta_1 \geq \frac{|J|}{4},$$

which implies

$$|A_3| \leq 2\pi\varphi\left(\frac{|J|}{4}\right). \tag{1.2.24}$$

To estimate A_2 we denote

$$a_k = \int_{\pi(k-\delta)/n}^{\pi(k+\delta)/n} \varphi(t)dt, \quad k \in \mathbb{Z}. \tag{1.2.25}$$

We have

$$a_0 = \int_{-\pi\delta/n}^{\pi\delta/n} \varphi(t)dt. \tag{1.2.26}$$

Using properties of φ we have $a_k = a_{-k}$ and $a_1 \geq a_2 \geq \ldots$. Using Chebishev's inequality we have $\varphi(t) \leq 1/t$. Thus we obtain

$$a_k \leq a_1 = \int_{\pi(1-\delta)/n}^{\pi(1+\delta)/n} \varphi(t)dt \leq \frac{2\pi\delta/n}{\pi(1-\delta)/n} = \frac{2\delta}{1-\delta} < 4\delta, \quad k \geq 1. \tag{1.2.27}$$

Using (1.2.20), (1.2.25), (1.2.26) and (1.2.27), we get

$$\begin{aligned}
A_2 &\geq \sum_{k=p}^{q-2} \int_{\pi(2k+1-\delta)/n}^{\pi(2k+1+\delta)/n} [\varphi(\theta_1 - t) - \varphi(\theta_2 - t)]dt - 8\delta \\
&= \sum_{k=p}^{q-1} \int_{\pi(2(m-k)-\delta)/n}^{\pi(2(m-k)+\delta)/n} \varphi(t)dt - \sum_{k=p}^{q-1} \int_{\pi(2(m-k)+1-\delta)/n}^{\pi(2(m-k)+1+\delta)/n} \varphi(t)dt - 8\delta \\
&= \sum_{k=m-q+1}^{m-p} a_{2k} - \sum_{k=m-q+1}^{m-p} a_{2k+1} - 8\delta \geq a_0 - a_1 - a_{-1} - 8\delta \\
&> \int_{-\pi\delta/n}^{\pi\delta/n} \varphi(t)dt - 16\delta.
\end{aligned}$$

Combining this with (1.2.23) and (1.2.24), we get

$$\int_{\mathbb{T}} \varphi(\theta_1 - t)\mathbb{I}_E(t)dt - \int_{\mathbb{T}} \varphi(\theta_2 - t)\mathbb{I}_E(t)dt \geq \int_{-\frac{\pi\delta}{n}}^{\frac{\pi\delta}{n}} \varphi(t)dt - 16\delta - 2\pi\varphi\left(\frac{|J|}{4}\right),$$

which together with (1.2.22) implies (1.2.19). To deduce the case $E \cap J = J \setminus U_n^\delta$ notice, that for the complement E^c we have $E^c \cap J = J \cap U_n^\delta$ and so (1.2.19) holds for E^c. Therefore we obtain

$$\begin{aligned}
&\operatorname{OSC}_{\theta\in\left[x-\frac{4\pi}{n}, x+\frac{4\pi}{n}\right]} \int_{\mathbb{T}} \varphi(\theta - t)\mathbb{I}_E(t)dt \\
&\qquad = \operatorname{OSC}_{\theta\in\left[x-\frac{4\pi}{n}, x+\frac{4\pi}{n}\right]} \left(\|\varphi\|_1 - \int_{\mathbb{T}} \varphi(\theta - t)\mathbb{I}_E(t)dt\right) \\
&\qquad = \operatorname{OSC}_{\theta\in\left[x-\frac{4\pi}{n}, x+\frac{4\pi}{n}\right]} \left(\int_{\mathbb{T}} \varphi(\theta - t)\mathbb{I}_{E^c}(t)dt\right) \\
&\qquad > \int_{-\frac{\pi\delta}{n}}^{\frac{\pi\delta}{n}} \varphi(t)dt - 16\delta - 2\pi\varphi\left(\frac{|J|}{4}\right),
\end{aligned}$$

which completes the proof of the lemma. □

1.3 The case of bounded measures and L^1

Proof of Theorem 1.1. Without loss of generality we may assume that $x_0 = 0$ and $\mu'(x_0) = 0$. We fix a function $\theta : (0, 1) \to \mathbb{R}$ with $|\theta(r)| \leq \lambda(r)$. From $\Pi(\lambda, \varphi) < \infty$ we get

$$|\theta(r)| \cdot \|\varphi_r\|_\infty \leq 2\Pi, \quad r_0 < r < 1. \tag{1.3.1}$$

Using the property *Φ2*, we may define a collection of numbers $\varepsilon_r > 0$ such that

$$\varepsilon_r \searrow 0, \quad \delta_r = \sup\{|t| : \varphi_r(t) \geq \varepsilon_r\} \to 0 \text{ as } r \to 1. \tag{1.3.2}$$

Applying Lemma 1.1, for any $0 < r < 1$ we define a family of intervals $I_j^{(r)}$, $j = 1, 2, \ldots, n_r$ such that

$$|I_j^{(r)}| \le 2\delta_r, \quad j = 1, 2, \ldots, n_r, \tag{1.3.3}$$

$$\sum_{j=1}^{n_r} \left| I_j^{(r)} \right| < 10\varepsilon_r^{-1} \max\{1, |\theta(r)| \cdot \|\varphi_r\|_\infty, \|\varphi_r\|_1\}, \tag{1.3.4}$$

$$\left| \varphi_r(\theta(r) - t) - \sum_{j=1}^{n_r} \varepsilon_j^{(r)} \mathbb{I}_{I_j^{(r)}}(t) \right| \le \varepsilon_r, \tag{1.3.5}$$

where $\varepsilon_j^{(r)} = \pm\varepsilon_r$. From (1.3.1) and (1.3.4) we conclude

$$\varepsilon_r \cdot \sum_{j=1}^{n_r} \left| I_j^{(r)} \right| \le L, \quad r_0 < r < 1, \tag{1.3.6}$$

where L is a positive constant. From (1.3.2) and (1.3.5) we obtain

$$\Phi_r(\theta(r), d\mu) = \int_{\mathbb{T}} \varphi_r(\theta(r) - t) d\mu(t) = \sum_{j=1}^{n_r} \varepsilon_j^{(r)} \int_{I_j^{(r)}} d\mu(t) + o(1), \tag{1.3.7}$$

where $o(1) \to 0$ as $r \to 1$. Using this, we get

$$|\Phi_r(\theta(r), d\mu)| \le \varepsilon_r \cdot \sum_{j=1}^{n_r} \left| I_j^{(r)} \right| \cdot \frac{1}{\left| I_j^{(r)} \right|} \left| \int_{I_j^{(r)}} d\mu(t) \right| + o(1). \tag{1.3.8}$$

According to (1.3.2) and (1.3.3), we have

$$\max_{1 \le j \le n_r} \frac{1}{\left| I_j^{(r)} \right|} \left| \int_{I_j^{(r)}} d\mu(t) \right| \to \mu'(0) = 0 \text{ as } r \to 1.$$

This together with (1.3.6) and (1.3.7) implies that $\Phi_r(\theta(r), d\mu) \to 0$ as $r \to 1$. □

Proof of Theorem 1.2. Let $\theta(r)$ satisfies (1.3.1). We again assume that $x_0 = 0$, $\mu'(x_0) = 0$ and so we will have $|\mu|'(0) = 0$. Then, repeating the same process of the proof of Theorem 1.1 at this time for the functions $\varphi_r^*(t)$ together with the measure $|\mu|$, instead of (1.3.7) we obtain

$$\int_{\mathbb{T}} \varphi_r^* \left(\theta(r) - t\right) d|\mu|(t) = \sum_{j=1}^{n_r} \varepsilon_j^{(r)} \int_{I_j^{(r)}} d|\mu|(t) + o(1).$$

Then we get

$$\begin{aligned} |\Phi_r(\theta(r), d\mu)| &\le \int_{\mathbb{T}} \varphi_r^*(\theta(r) - t) d|\mu|(t) \\ &= \varepsilon_r \cdot \sum_{j=1}^{n_r} |I_j^{(r)}| \cdot \frac{1}{|I_j^{(r)}|} \int_{I_j^{(r)}} d|\mu|(t) + o(1). \end{aligned}$$

Since $|\mu|$ is differentiable at 0, we get

$$\Phi_r(\theta(r), d\mu) \to 0.$$

□

Proof of Theorem 1.3. For any $0 < r < 1$ there exist a point $x_r \in \mathbb{T}$, a number $0 < \delta_r < \frac{1}{4}\lambda(r)$ and a measurable set $E_r \subset \mathbb{T}$ such that

$$E_r \subset (x_r - \delta_r, x_r + \delta_r), \quad |E_r| > \frac{3\delta_r}{2}, \tag{1.3.9}$$

$$|\varphi_r(x)| > \frac{\|\varphi_r\|_\infty}{2}, \quad x \in E_r. \tag{1.3.10}$$

From these relations it follows that $\varphi_r^*(x) > \frac{1}{2}\|\varphi_r\|_\infty$ if $x \in (-|x_r|, |x_r|)$. On the other hand, by property *Φ3* we have $\|\varphi_r^*\|_1 \leq C_\varphi$, which imples

$$|x_r| \leq \frac{2C_\varphi}{\|\varphi_r\|_\infty}, \quad 0 < r < 1. \tag{1.3.11}$$

Denote

$$n(r) = \left[\frac{4\pi}{\lambda(r)}\right] \in \mathbb{N}, \tag{1.3.12}$$

$$\Delta_r = \bigcup_{k=0}^{n(r)-1} \left[\frac{2\pi k}{n(r)} - \delta_r, \frac{2\pi k}{n(r)} + \delta_r\right]. \tag{1.3.13}$$

If $x \in \mathbb{T}$ is an arbitrary point, then

$$x \in \left[\frac{2\pi k_0}{n(r)}, \frac{2\pi(k_0+1)}{n(r)}\right)$$

for some $k_0 \in \{0, 1, \ldots, n(r) - 1\}$. Consider the function

$$f_r(x) = \frac{\mathbb{I}_{\Delta_r}(x)}{|\Delta_r|} \operatorname{sgn} \varphi_r\left(\frac{2\pi k_0}{n(r)} + x_r - x\right). \tag{1.3.14}$$

Clearly $\|f_r\|_1 = 1$. Taking $\theta = x - x_r - \frac{2\pi k_0}{n(r)}$, from (1.3.11) and (1.3.12) we obtain

$$|\theta| < \frac{2\pi}{n(r)} + |x_r| < \frac{2\pi\lambda(r)}{4\pi - \lambda(r)} + \frac{2C_\varphi}{\|\varphi_r\|_\infty} \leq \lambda(r)\left(\frac{1}{2} + \frac{2C_\varphi}{\lambda(r)\|\varphi_r\|_\infty}\right). \tag{1.3.15}$$

Using the condition $\Pi(\lambda,\varphi)=\infty$ and Lemma 1.2 we may fix a sequence $r_k \nearrow 1$ such that

$$\lambda(r_k)\|\varphi_{r_k}\|_\infty > C_\varphi \cdot 2^{k+3}\left(1+k+\max_{1\le j<k}\frac{1}{|\Delta_{r_j}|}\right), \quad k=1,2,\ldots, \tag{1.3.16}$$

$$\sup_{\theta\in\mathbb{T}}\frac{1}{|\Delta_{r_j}|}\int_{\Delta_{r_j}}\varphi^*_{r_k}(\theta-t)\,dt \le C_\varphi, \quad k=1,2,\ldots,j-1. \tag{1.3.17}$$

From (1.3.15) and (1.3.16) we conclude

$$|\theta|<\lambda(r), \text{ if } r=r_k. \tag{1.3.18}$$

Using (1.3.10), (1.3.12) and (1.3.13), for the same x we get

$$\begin{aligned}
\Phi_r(x-\theta,f_r) &= \int_{\mathbb{T}}\varphi_r\left(\frac{2\pi k_0}{n(r)}+x_r-t\right)f_r(t)dt\\
&= \frac{1}{|\Delta_r|}\int_{\Delta_r}\left|\varphi_r\left(\frac{2\pi k_0}{n(r)}+x_r-t\right)\right|dt\\
&\ge \frac{1}{2\delta_r n(r)}\int_{2\pi k_0/n(r)-\delta_r}^{2\pi k_0/n(r)+\delta_r}\left|\varphi_r\left(\frac{2\pi k_0}{n(r)}+x_r-t\right)\right|dt\\
&= \frac{1}{2\delta_r n(r)}\int_{x_r-\delta_r}^{x_r+\delta_r}|\varphi_r(u)|\,du\\
&\ge \frac{1}{2\delta_r n(r)}\cdot\frac{3\delta_r}{2}\cdot\frac{\|\varphi_r\|_\infty}{2}\ge\frac{3\lambda(r)\|\varphi_r\|_\infty}{16}.
\end{aligned} \tag{1.3.19}$$

Define

$$f(x)=\sum_{k=1}^\infty 2^{-k}f_{r_k}(x)\in L^1(\mathbb{T}),$$

and show that

$$\lim_{k\to\infty}\sup_{\theta\in\lambda(r_k,x)}\Phi_{r_k}(\theta,f)=\infty.$$

We split $\Phi_{r_k}(\theta,f)$ in the following way

$$\begin{aligned}
\Phi_{r_k}(\theta,f) &= \sum_{j=1}^\infty 2^{-j}\Phi_{r_k}(\theta,f_{r_j})\\
&= \sum_{j=1}^{k-1}2^{-j}\Phi_{r_k}(\theta,f_{r_j})+2^{-k}\Phi_{r_k}(\theta,f_{r_k})+\sum_{j=k+1}^\infty 2^{-j}\Phi_{r_k}(\theta,f_{r_j})\\
&= S^1+S^2+S^3
\end{aligned} \tag{1.3.20}$$

From (1.3.16), (1.3.18) and (1.3.19) it follows that

$$\sup_{\theta\in\lambda(r_k,x)}S^2=\sup_{\theta\in\lambda(r_k,x)}2^{-k}\Phi_{r_k}(\theta,f_{r_k})\ge C_\varphi\left(1+k+\max_{1\le j<k}\frac{1}{|\Delta_{r_j}|}\right) \tag{1.3.21}$$

Furthermore, using (1.3.14) and *Φ3* propery of $\{\varphi_r\}$ we get

$$\begin{aligned}\sup_{\theta\in\lambda(r_k,x)}|S^1| &= \sup_{\theta\in\lambda(r_k,x)}\left|\sum_{j=1}^{k-1}2^{-j}\int_{\mathbb{T}}\varphi_{r_k}(\theta-t)f_{r_j}(t)\,dt\right| \\ &\le \sup_{\theta\in\lambda(r_k,x)}\sum_{j=1}^{k-1}\frac{2^{-j}}{|\Delta_{r_j}|}\int_{\Delta_{r_j}}|\varphi_{r_k}(\theta-t)|\,dt \\ &\le \sum_{j=1}^{k-1}\frac{2^{-j}}{|\Delta_{r_j}|}\int_{\mathbb{T}}\varphi^*_{r_k}(u)\,du \le C_\varphi\cdot\max_{1\le j<k}\frac{1}{|\Delta_{r_j}|}.\end{aligned} \tag{1.3.22}$$

Finally, using (1.3.17) we get

$$\begin{aligned}\sup_{\theta\in\lambda(r_k,x)}|S^3| &= \sup_{\theta\in\lambda(r_k,x)}\left|\sum_{j=k+1}^{\infty}2^{-j}\int_{\mathbb{T}}\varphi_{r_k}(\theta-t)f_{r_j}(t)\,dt\right| \\ &\le \sup_{\theta\in\lambda(r_k,x)}\sum_{j=k+1}^{\infty}2^{-j}\cdot\frac{1}{|\Delta_{r_j}|}\int_{\Delta_{r_j}}|\varphi_{r_k}(\theta-t)|\,dt \\ &\le \sum_{j=k+1}^{\infty}2^{-j}\cdot\sup_{\theta\in\mathbb{T}}\frac{1}{|\Delta_{r_j}|}\int_{\Delta_{r_j}}\varphi^*_{r_k}(\theta-t)\,dt\le C_\varphi\end{aligned} \tag{1.3.23}$$

So, from (1.3.22), (1.3.21), (1.3.23) and (1.3.20) it follows

$$\sup_{\theta\in\lambda(r_k,x)}\Phi_{r_k}(\theta,f)\ge\sup_{\theta\in\lambda(r_k,x)}S^2-\sup_{\theta\in\lambda(r_k,x)}|S^1|-\sup_{\theta\in\lambda(r_k,x)}|S^3|\ge C_\varphi\cdot k,$$

which imples (1.1.3). □

Proof of Theorem 1.4. Without loss of generality we may assume that f is non-negative. Let $x,y\in\mathbb{T}$, $0<r<1$ and $|x-y|<\lambda(r)$. We split the integral $\Phi_r(y,f)$ as follows

$$\begin{aligned}\Phi_r(y,f) &= \int_{\mathbb{T}}\varphi_r(t)f(y-t)\,dt \\ &= \int_{|t|\le\mu(r)}\varphi_r(t)f(y-t)\,dt \\ &+ \int_{\mu(r)<|t|<\lambda(r)}\varphi_r(t)f(y-t)\,dt \\ &+ \int_{\lambda(r)\le|t|\le\pi}\varphi_r(t)f(y-t)\,dt = I^1+I^2+I^3.\end{aligned} \tag{1.3.24}$$

First of all, from Lemma 1.3 we have

$$\sup_{\substack{|x-y|<\lambda(r)\\0<r<1}}|I^3|\le 8C_\varphi\cdot Mf(x). \tag{1.3.25}$$

Notice that from the condition $\tilde{\Pi}_p(\lambda, \varphi) < \infty$ it follows that

$$\lambda(r) \leq \tilde{\Pi}_p \cdot \mu(r)\varphi_*^{-p}(r).$$

Hence, from Lemma 1.5 we get

$$\sup_{\substack{|x-y|<\lambda(r)\\ 0<r<1}} |I^2| \leq \frac{4\tilde{\Pi}_p^{1/p}}{2^{1/p}-1} \left(Mf^p(x)\right)^{1/p}. \tag{1.3.26}$$

Furthermore, using the definition of $\mu(r)$, for I_1 we obtain

$$\begin{aligned} |I^1| &\leq \int_{|t|\leq\mu(r)} \varphi_r^*(t) f(y-t) \\ &\leq \|\varphi_r\|_\infty \int_{-\mu(r)}^{\mu(r)} f(y-t)\, dt \\ &= 2\mu(r)\|\varphi_r\|_\infty m_f(y,\mu(r)) = 2\varphi_*(r) m_f(y,\mu(r)), \end{aligned}$$

where

$$m_f(y,t) = \frac{1}{2t}\int_{y-t}^{y+t} |f(u)|\, du, \quad y \in \mathbb{T},\, t > 0.$$

To estimate I_1 we split the supremum into two parts as we did in Lemma 1.5:

$$\begin{aligned} \sup_{\substack{|x-y|\leq\mu(r)\leq\lambda(r)\\ 0<r<1}} I^1 \leq & \sup_{\substack{|x-y|\leq\mu(r)\leq\lambda(r)\\ 0<r<1}} 2\varphi_*(r) m_f(y,\mu(r)) \\ & + \sup_{\substack{\mu(r)<|x-y|<\lambda(r)\\ 0<r<1}} 2\varphi_*(r) m_f(y,\mu(r)) \end{aligned} \tag{1.3.27}$$

Notice that the second supremum is $T_1 f(x)$, which can be estimated due to Lemma 1.4. To estimate the first one, note that $\mu(r) \leq \lambda(r) \leq \tilde{\Pi}_p \mu(r)\varphi_*(r)^{-p}$ implies

$$\varphi_*(r) \leq \tilde{\Pi}_p^{1/p}. \tag{1.3.28}$$

On the other hand, from $|x-y| \leq \mu(r)$ implies $m_f(y,\mu(r)) \leq Mf(x)$, which together with (1.3.27), (1.3.28) and Lemma 1.4 gives

$$\sup_{\substack{|x-y|\leq\mu(r)\leq\lambda(r)\\ 0<r<1}} |I^1| \leq 2\tilde{\Pi}_p^{1/p} \left(Mf^p(x)\right)^{1/p}. \tag{1.3.29}$$

Then, combining (1.3.25), (1.3.26), (1.3.29) and (1.3.24), we get

$$\sup_{\substack{|x-y|<\lambda(r)\\ 0<r<1}} \Phi_r(y,f) \leq \left(2\tilde{\Pi}_p^{1/p} + \frac{4\tilde{\Pi}_p^{1/p}}{2^{1/p}-1} + 8C_\varphi\right) \left(Mf^p(x)\right)^{1/p},$$

which implies (1.1.5).

To get weak type inequality for Φ^*_λ, note that

$$\begin{aligned}|\{x \in \mathbb{T}\colon \Phi^*_\lambda(x,f) > t\}| &= |\{x \in \mathbb{T}\colon\ (\Phi^*_\lambda(x,f))^p > t^p\}| \\ &\le |\{x \in \mathbb{T}\colon Mf^p(x) > t^p/C^p\}| \\ &\le \frac{C_M C^p}{t^p}\|f^p\|_1 = \frac{C_M C^p}{t^p}\|f\|_p^p,\end{aligned}$$

where $C_M = \|M\|_{L^1\to L^{1,w}}$ is the weak $(1,1)$ norm of the maximal operator M. □

1.4 The case of L^∞

Proof of Theorem 1.5. Since $\Pi_\infty = 0$, then for any $0 < \varepsilon < 1/2$ we may chose $\delta > 0$ and $0 < \tau < 1$, such that

$$\int_{-\delta\lambda(r)}^{\delta\lambda(r)} \varphi_r(t)dt < \varepsilon, \quad \tau < r < 1. \tag{1.4.1}$$

Then we define

$$\varphi_r^{(1)}(x) = \begin{cases} \varphi_r(x) - \varphi_r(\delta\lambda(r)) & \text{if} \quad |x| \le \delta\lambda(r), \\ 0 & \text{if} \quad \delta\lambda(r) < |x| < \pi. \end{cases}$$

and

$$\varphi_r^{(2)}(x) = \frac{\varphi_r(x) - \varphi_r^{(1)}(x)}{\|\varphi_r(x) - \varphi_r^{(1)}\|_{L^1}} = \frac{\varphi_r(x) - \varphi_r^{(1)}(x)}{1 - l_r}$$

where

$$l_r = \int_{-\delta\lambda(r)}^{\delta\lambda(r)} (\varphi_r(t) - \varphi_r(\delta\lambda(r))\,dt < \varepsilon < \frac{1}{2}, \quad \tau < r < 1.$$

It is clear, that $\{\varphi_r^{(2)}\}$ is a regular approximate identity and we have

$$\varphi_r(x) = \varphi_r^{(1)}(x) + (1 - l_r)\varphi_r^{(2)}(x). \tag{1.4.2}$$

From (1.4.1) it follows that

$$\left|\int_{\mathbb{T}} \varphi_r^{(1)}(x-t)f(t)dt\right| \le \|f\|_\infty \int_{-\delta\lambda(r)}^{\delta\lambda(r)} \varphi_r(t)dt \le \varepsilon\|f\|_\infty \tag{1.4.3}$$

and

$$\varphi_r(\delta\lambda(r)) \cdot 2\delta\lambda(r) < \varepsilon, \quad \tau < r < 1.$$

Thus, using the definition of $\varphi_r^{(2)}(x)$, we get

$$\|\varphi_r^{(2)}\|_\infty \cdot \lambda(r) < \frac{\varepsilon}{2\delta(1-l_r)} < \frac{\varepsilon}{4\delta}$$

Using this and Theorem 1.1 we conclude, that

$$\lim_{\substack{r\to 1\\ y\in\lambda(r,x)}} \int_{\mathbb{T}} \varphi_r^{(2)}(y-t)f(t)dt = f(x) \tag{1.4.4}$$

at any Lebesgue point. Now without loss of generality we assume that $f(x) \geq 0$. If x is an arbitrary Lebesgue point, using (1.4.2), (1.4.3) and (1.4.4) we get

$$\limsup_{\substack{r\to 1\\ y\in\lambda(r,x)}} \Phi_r(y,f) \leq \varepsilon\|f\|_\infty + f(x),$$

$$\liminf_{\substack{r\to 1\\ y\in\lambda(r,x)}} \Phi_r(y,f) \geq -\varepsilon\|f\|_\infty + (1-\varepsilon)f(x).$$

Since ε can be taken sufficiently small, we get

$$\lim_{\substack{r\to 1\\ y\in\lambda(r,x)}} \Phi_r(y,f) = f(x),$$

and the theorem is proved. □

Proof of Theorem 1.6. Since $\Pi_\infty > 0$, there exist sequences $\delta_k \searrow 0$ and $r_k \to 1$, such that

$$\int_{-\delta_k\lambda(r_k)}^{\delta_k\lambda(r_k)} \varphi_{r_k}(t)dt > \frac{\Pi_\infty}{2}, \quad k = 1,2,\ldots. \tag{1.4.5}$$

Denote

$$U_k = U_{n_k}^{\delta_k}, \quad n_k = \left[\frac{\pi}{\lambda(r_k)}\right], \tag{1.4.6}$$

where U_n^δ is defined in the Lemma 1.9. Define the sequences of measurable sets E_n by

$$E_1 = U_1, \quad E_k = E_{k-1} \triangle U_k = (E_{k-1}\setminus U_k)\cup(U_k\setminus E_{k-1}), \quad k>1$$

We say J is an adjacent interval for E_k, if it is a maximal interval containing either in E_k or $(E_k)^c$. The family of all this intervals form a covering of whole $\mathbb{T}$. It is easy to observe, that a suitable selection of δ_k and r_k may provide

$$\varphi_{r_k}\left(\frac{|J|}{4}\right) < \frac{\Pi_\infty}{16\pi}, \text{ if } J \text{ is adjacent for } E_{k-1}, \tag{1.4.7}$$

$$\delta_j \leq \frac{\Pi_\infty}{2^{j+5}\|\varphi_{r_k}\|_\infty}, \quad j \geq k+1, \tag{1.4.8}$$

It is easy to observe, that if $k < m$, then

$$\|\mathbb{I}_{E_k} - \mathbb{I}_{E_m}\|_1 = |E_k \triangle E_m| \leq \sum_{j\geq k+1} |U_j| \tag{1.4.9}$$

This implies, that $\mathbb{I}_{E_n}$ converges to a function $f \in L^1$. Using Egorov's theorem, we conclude that $f = \mathbb{I}_E$ for some measurable set $E \subset \mathbb{T}$. Tending m to infinity, from (1.4.8) and (1.4.9) we get

$$|E_k \triangle E| \leq \left| \bigcup_{j\geq k+1} U_j \right| \leq 2\pi \sum_{j\geq k+1} \delta_j \leq \frac{\Pi_\infty}{16\|\varphi_{r_k}\|_\infty}. \tag{1.4.10}$$

Fix a point $x \in \mathbb{T}$. We have $x \in J$ where J is an adjacent interval for E_{k-1}. From the definition of E_k it follows that either

$$E_k \cap J = J \cap U_k \quad \text{or} \quad E_k \cap J = J \setminus U_k.$$

From (1.4.6) we have

$$\lambda(r_k, x) = (x - \lambda(r_k), x + \lambda(r_k)) \subset \left[x - \frac{4\pi}{n_k}, x + \frac{4\pi}{n_k} \right].$$

Thus, applying Lemma 1.9, (1.4.5) and (1.4.7), we get

$$\begin{aligned}
&\operatorname{OSC}_{\theta\in\lambda(r_k,x)} \Phi_{r_k}(\theta, \mathbb{I}_{E_k}) \\
&\qquad \geq \operatorname{OSC}_{\theta\in\left[x-\frac{4\pi}{n_k}, x+\frac{4\pi}{n_k}\right]} \Phi_{r_k}(\theta, \mathbb{I}_{E_k}) \\
&\qquad \geq \int_{-\frac{\pi\delta_k}{n_k}}^{\frac{\pi\delta_k}{n_k}} \varphi_{r_k}(t)dt - 16\delta_k - 2\pi\varphi_{r_k}\left(\frac{|J|}{4}\right) \\
&\qquad \geq \int_{-\delta_k\lambda(r_k)}^{\delta_k\lambda(r_k)} \varphi_{r_k}(t)dt - 16\delta_k - \frac{\Pi_\infty}{8} \\
&\qquad \geq \frac{\Pi_\infty}{4} - 16\delta_k,
\end{aligned}$$

where

$$\operatorname{OSC}_{x\in E} f(x) = \sup_{x,y\in E} |f(x) - f(y)|.$$

From (1.4.10) we conclude

$$\begin{aligned}
&\operatorname{OSC}_{\theta\in\lambda(r_k,x)} \Phi_{r_k}(\theta, \mathbb{I}_E) \\
&\qquad > \operatorname{OSC}_{\theta\in\lambda(r_k,x)} \Phi_{r_k}(\theta, \mathbb{I}_{E_k}) - \frac{\Pi_\infty}{16} \geq \frac{\Pi_\infty}{8} - 16\delta_k,
\end{aligned}$$

which completes the proof of the theorem since $\delta_k \to 0$. □

CHAPTER 2

Littlewood type theorems

2.1 Introduction

In this chapter we generalize Littlewood's theorem for the integrals with general kernels. Here we remind Littlewood's theorem as well as generalized versions for Blaschke products and harmonic functions.

Theorem F (Littlewood, 1927). *If a continuous function* $\lambda(r) : [0,1] \to \mathbb{R}$ *satisfies the conditions*

$$\lambda(1) = 0, \quad \lim_{r\to 1} \frac{\lambda(r)}{1-r} = \infty, \tag{2.1.1}$$

then there exists a bounded analytic function $f(z)$, $z \in D$, *such that the boundary limit*

$$\lim_{r\to 1} f\left(re^{i(x+\lambda(r))}\right)$$

does not exist almost everywhere on $\mathbb{T}$.

Theorem G (Lohwater and Piranian, 1957). *If a continuous function* $\lambda(r)$ *satisfies (2.1.1), then there exists a Blaschke product* $B(z)$ *such that the limit*

$$\lim_{r\to 1} B\left(re^{i(x+\lambda(r))}\right)$$

does not exist for any $x \in \mathbb{T}$.

Theorem H (Aikawa, 1990). *If* $\lambda(r)$ *is continuous and satisfies the condition (2.1.1), then there exists a bounded harmonic function* $u(z)$ *on the unit disc, such that the limit*

$$\lim_{r\to 1} u\left(re^{i(x+\lambda(r))}\right)$$

does not exist for any $x \in \mathbb{T}$.

In Section 2.2 we construct a characteristic function with Littlewood type divergence property for general kernels:

Theorem 2.1. *Let $\{\varphi_r\}$ be a family of kernels with $\Phi 1$, $\Phi 4$, $\Phi 5$. If a function $\lambda \in C[0,1]$ satisfies the conditions $\lambda(1) = 0$ and*

$$\Pi^*(\lambda, \varphi) = \limsup_{\delta \to 0} \liminf_{r \to 1} \int_{-\delta\lambda(r)}^{\delta\lambda(r)} \varphi_r(t)dt > \frac{1}{2},$$

then there exists a measurable set $E \subset \mathbb{T}$ such that

$$\limsup_{r \to 1} \Phi_r \left(x + \lambda(r), \mathbb{I}_E\right) - \liminf_{r \to 1} \Phi_r \left(x + \lambda(r), \mathbb{I}_E\right) \geq 2\Pi^* - 1.$$

In Section 2.3 we construct a Blaschke product with Littlewood type divergence property for general kernels:

Theorem 2.2. *Let a family of kernels $\{\varphi_r\}$ satisfies $\Phi 1$, $\Phi 4$, $\Phi 5$ and for $\lambda \in C[0,1]$ we have $\lambda(1) = 0$ and $\Pi^*(\lambda, \varphi) = 1$. Then there exists a function $B \in L^\infty(\mathbb{T})$, which is the boundary function of a Blaschke product, such that the limit*

$$\lim_{r \to 1} \Phi_r \left(x + \lambda(r), B\right)$$

does not exist for any $x \in \mathbb{T}$.

2.2 Divergence with characteristic function

We consider the sets

$$U(n, \delta) = \bigcup_{j=0}^{n-1} \left(\frac{\pi(2j - \delta)}{n}, \frac{\pi(2j + \delta)}{n} \right) \subset \mathbb{T}, \tag{2.2.1}$$

which will be used in the proofs of both theorems.

Proof of Theorem 2.1. Using the definition of Π^* and the absolute continuity property *Φ5*, we may choose numbers δ_k, u_k, v_k $(k \in \mathbb{N})$, satisfying

$$\delta_k < 2^{-k-5}, \quad 1 > v_k > u_k \to 1, \quad 3\lambda(v_k) \le \lambda(u_k) < \pi, \tag{2.2.2}$$

$$\int_{-\delta_k\lambda(u_k)}^{\delta_k\lambda(u_k)} \varphi_{u_k}(t)dt > \Pi^* \cdot (1 - 2^{-k}), \quad k = 1, 2, \dots, \tag{2.2.3}$$

$$\int_e |\varphi_r(t)|dt < 2^{-k}, \tag{2.2.4}$$

where the last bound holds whenever

$$0 < r < v_k, \quad |e| \le 10\pi \sum_{j \ge k+1} \sqrt[4]{\delta_j}. \tag{2.2.5}$$

We will consider the same sequences (2.2.2) with properties (2.2.3)−(2.2.5) in the proof of Theorem 2.2 as well. We note that $\sqrt[4]{\delta_j}$ in (2.2.5) is necessary only in the proof of Theorem 2.2, but for Theorem 2.1 just δ_j is enough. Denote

$$U_k = U(n_k, 5\delta_k), \quad n_k = \left[\frac{5\pi}{\lambda(u_k)}\right], \quad k \in \mathbb{N}, \tag{2.2.6}$$

and define the sequence of measurable sets $E_k \subset \mathbb{T}$ by

$$E_1 = U_1, \tag{2.2.7}$$

$$E_k = \begin{cases} E_{k-1} \setminus U_k & \text{if } k \text{ is even}, \\ E_{k-1} \cup U_k & \text{if } k \text{ is odd}. \end{cases} \tag{2.2.8}$$

It is easy to observe, that if $k < m$, then

$$\|\mathbb{I}_{E_k} - \mathbb{I}_{E_m}\|_1 = |E_k \triangle E_m| \le \sum_{j \ge k+1} |U_j|. \tag{2.2.9}$$

This implies that $\mathbb{I}_{E_n}$ converges to a function f in L^1 norm. Using Egorov's theorem, we conclude that $f = \mathbb{I}_E$ for some measurable set $E \subset \mathbb{T}$. Tending m to infinity, from (2.2.9) we get

$$|E \triangle E_k| = |(E \setminus E_k) \cup (E_k \setminus E)| \le \sum_{j \ge k+1} |U_j| \le 10\pi \sum_{j \ge k+1} \delta_j. \tag{2.2.10}$$

Take an arbitrary $x \in \mathbb{T}$. There exists an integer $1 \le j_0 \le n_k$ such that

$$\frac{2\pi j_0}{n_k} - x \in \left[\frac{2\pi}{n_k}, \frac{4\pi}{n_k}\right] \subset \left[\frac{\lambda(u_k)}{3}, \lambda(u_k)\right] \subset [\lambda(v_k), \lambda(u_k)]$$

and therefore, since $\lambda(r)$ is continuous, we may find a number r, $u_k \le r \le v_k$, such that

$$\lambda(r) = \frac{2\pi j_0}{n_k} - x. \tag{2.2.11}$$

If $k \in \mathbb{N}$ is odd, then according to the definition of E_k we get

$$E_k \supset U_k \supset I = \left(\frac{\pi(2j_0 + 5\delta_k)}{n_k}, \frac{\pi(2j_0 - 5\delta_k)}{n_k} \right).$$

Thus, using (2.2.3), (2.2.11) as well as the definition of n_k from (2.2.6), we conclude

$$\begin{aligned} \Phi_r(x + \lambda(r), \mathbb{I}_{E_k}) &\ge \int_I \varphi_r(x + \lambda(r) - t)dt \\ &= \int_I \varphi_r \left(\frac{2\pi j_0}{n_k} - t \right) dt \\ &= \int_{-5\pi\delta_k/n_k}^{5\pi\delta_k/n_k} \varphi_r(t)\, dt \\ &\ge \int_{-\delta_k \lambda(u_k)}^{\delta_k \lambda(u_k)} \varphi_r(t)dt > \Pi^* \cdot (1 - 2^{-k}). \end{aligned} \tag{2.2.12}$$

From (2.2.4) and (2.2.10) it follows that

$$|\Phi_r(t, \mathbb{I}_E) - \Phi_r(t, \mathbb{I}_{E_k})| < 2^{-k}, \quad t \in \mathbb{T}, \quad 0 < r < v_k,$$

and hence from (2.2.12) we obtain

$$\limsup_{r \to 1} \Phi_r(x + \lambda(r), \mathbb{I}_E) \ge \Pi^*. \tag{2.2.13}$$

If $k \in \mathbb{N}$ is even, then we have $E_k \cap U_k = \varnothing$ and therefore $E_k \cap I = \varnothing$. Thus we get

$$\begin{aligned} \Phi_r(x + \lambda(r), \mathbb{I}_{E_k}) &\le \int_{\mathbb{T}} \varphi_r(x + \lambda(r) - t)dt - \int_I \varphi_r(x + \lambda(r) - t)dt \\ &\le 1 - \int_{-\delta_k \lambda(u_k)}^{\delta_k \lambda(u_k)} \varphi_r(t)\, dt \le 1 - \Pi^*(1 - 2^{-k}) \end{aligned}$$

and similarly we get

$$\liminf_{r \to 1} \Phi_r(x + \lambda(r), \mathbb{I}_E) \le 1 - \Pi^*. \tag{2.2.14}$$

Relations (2.2.13) and (2.2.14) complete the proof of the theorem. □

2.3 Divergence with Blaschke product

The following finite Blaschke products

$$b(n,\delta,z)=\frac{z^n-\rho^n}{\rho^n z^n-1}=\prod_{k=0}^{n-1}\frac{z-\rho e^{\frac{2\pi ik}{n}}}{\rho e^{\frac{2\pi ik}{n}}z-1},\quad \rho=e^{-\sqrt{\delta}/n}. \tag{2.3.1}$$

play significant role in the proof of Theorem 2.2. Similar products are used in the proof of theorem Theorem G too. If $z=e^{ix}$, then (2.3.1) defines a continuous function in $H^\infty(\mathbb{T})$. We will use the set $U(n,\delta)$ defined in (2.2.1). The following lemma shows that on $U(n,\delta)$ the function (2.3.1) is approximative -1, and outside of $U(n,\sqrt[4]{\delta})$ is approximative 1.

Lemma 2.1. *There exists an absolute constant $C>0$ such that*

$$\left|b\left(n,\delta,e^{ix}\right)+1\right|\le C\sqrt{\delta},\quad x\in U(n,\delta), \tag{2.3.2}$$

$$\left|b\left(n,\delta,e^{ix}\right)-1\right|\le C\sqrt[4]{\delta},\quad x\in\mathbb{T}\setminus U(n,\sqrt[4]{\delta}). \tag{2.3.3}$$

Proof. Deduction of these inequalities based on the inequalities

$$\frac{|x|}{2}\le|e^{ix}-1|\le 2|x|,\quad \text{if}\quad |x|\le\pi.$$

If $x\in U(n,\delta)$, then we have

$$\begin{aligned}\left|b\left(n,\delta,e^{ix}\right)+1\right|&=\left|\frac{(e^{inx}-1)(\rho^n+1)}{\rho^n e^{inx}-1}\right|\le\frac{4\pi\delta}{1-e^{-\sqrt{\delta}}},\\&\le\frac{4e\pi\delta}{e^{\sqrt{\delta}}-1}\le\frac{8e\pi\delta}{\sqrt{\delta}}\le C\sqrt{\delta}.\end{aligned} \tag{2.3.4}$$

If $x\in\mathbb{T}\setminus U(n,\sqrt[4]{\delta})$, then $e^{inx}=e^{i\alpha}$ with $\pi\sqrt[4]{\delta}<|\alpha|<\pi$. Thus we obtain

$$\begin{aligned}\left|b\left(n,\delta,e^{ix}\right)-1\right|&=\left|\frac{(e^{inx}+1)(1-\rho^n)}{\rho^n e^{inx}-1}\right|=\frac{2(e^{\sqrt{\delta}}-1)}{|e^{inx}-e^{\sqrt{\delta}}|}\\&\le\frac{4\sqrt{\delta}}{|e^{inx}-1|-|e^{\sqrt{\delta}}-1|}\le\frac{4\sqrt{\delta}}{\pi\sqrt[4]{\delta}/2-2\sqrt{\delta}}\le C\sqrt[4]{\delta}.\end{aligned} \tag{2.3.5}$$

□

Proof of Theorem 2.2. First we choose numbers δ_k, u_k, v_k ($k\in\mathbb{N}$), satisfying (2.2.2)−(2.2.4) with $\Pi^*=1$. Then we denote

$$b_k(x)=b(n_k,\delta_k,e^{ix}),\quad n_k=\left[\frac{6\pi}{\lambda(u_k)}\right],\quad k\in\mathbb{N}, \tag{2.3.6}$$

and

$$B_k(x) = \prod_{j=1}^{k} b_j(x), \quad B(x) = \prod_{j=1}^{\infty} b_j(x).$$

The convergence of the infinite product follows from the bound (2.3.9), which will be obtained bellow. Observe that in the process of selection of the numbers (2.2.2) we were free to define $\delta_k > 0$ as small as needed. Besides, taking u_k to be close to 1 we may get n_k as big as needed. Using these notations and Lemma 2.1, aside of the conditions (2.2.2)$-$(2.2.4) we can additionally claim the bounds

$$\omega\left(2\pi/n_k, B_{k-1}\right) = \sup_{|x-x'|<2\pi/n_k} |B_{k-1}(x) - B_{k-1}(x')| < 2^{-k}, \tag{2.3.7}$$

$$|b_k(x) + 1| < 2^{-k}, \quad x \in U(n_k, 6\delta_k), \tag{2.3.8}$$

$$|b_k(x) - 1| < 2^{-k}, x \in \mathbb{T} \setminus U(n_k, \sqrt[4]{\delta_k}). \tag{2.3.9}$$

From (2.3.9) we get

$$\begin{aligned} |B(x) - B_k(x)| &= \left| \prod_{j\geq k+1} b_j(x) - 1 \right| \\ &\leq \prod_{j\geq k+1} (1 + 2^{-j}) - 1 < 2^{-k+1}, \quad x \in \mathbb{T} \setminus \bigcup_{j\geq k+1} U\left(n_j, \sqrt[4]{\delta_j}\right). \end{aligned} \tag{2.3.10}$$

Take an arbitrary $x \in \mathbb{T}$. There exists an integer $1 \leq j_0 \leq n_k$ such that

$$\frac{2\pi j_0}{n_k} - x \in \left[\frac{2\pi}{n_k}, \frac{4\pi}{n_k}\right] \subset \left[\frac{2\pi}{n_k}, \frac{5\pi}{n_k}\right] \subset \left[\frac{\lambda(u_k)}{3}, \lambda(u_k)\right] \subset [\lambda(v_k), \lambda(u_k)],$$

where the inclusions follow from the definition of n_k (see (2.3.6)) and from the inequality $3\lambda(v_k) \leq \lambda(u_k) < \pi$ coming from (2.2.2). Thus, since $\lambda(r)$ is continuous, we may find numbers $u_k \leq r' \leq r'' \leq v_k$, such that

$$\lambda(r') = \frac{2\pi j_0}{n_k} - x, \quad \lambda(r'') = \frac{2\pi j_0}{n_k} + \frac{\pi}{n_k} - x. \tag{2.3.11}$$

For the set

$$e = \bigcup_{j\geq k+1} U\left(n_j, \sqrt[4]{\delta_j}\right),$$

we have

$$|e| = 10\pi \sum_{j\geq k+1} \sqrt[4]{\delta_j}.$$

So taking $r \in [u_k, v_k]$, from (2.2.4) and (2.3.10) we conclude

$$\begin{aligned} &\left|\Phi_r(x,B) - \Phi_r(x,B_k)\right| \\ &\qquad \le \int_e \varphi_r(x-t)|B(t)-B_k(t)|dt + 2^{-k+1}\int_{\mathbb{T}\setminus e} \varphi_r(x-t)dt \\ &\qquad \le 2\cdot 2^{-k} + 2^{-k+1} = 4\cdot 2^{-k}, \quad x\in\mathbb{T}. \end{aligned} \tag{2.3.12}$$

If

$$t \in I = (-\delta_k\lambda(u_k), \delta_k\lambda(u_k)) \subset \left(-\frac{6\pi\delta_k}{n_k}, \frac{6\pi\delta_k}{n_k}\right),$$

then we have

$$\begin{aligned} &\frac{2\pi j_0}{n_k} - t \in U(n_k, 6\delta_k), \\ &\frac{2\pi j_0}{n_k} + \frac{\pi}{n_k} - t \in \mathbb{T}\setminus U(n_k, \sqrt[4]{\delta_k}). \end{aligned}$$

Then, using these relations, (2.3.8) and (2.3.7), we get

$$\begin{aligned} &\left|B_k\left(\frac{2\pi j_0}{n_k} - t\right) + B_{k-1}\left(\frac{2\pi j_0}{n_k}\right)\right| \\ &\qquad \le \left|B_{k-1}\left(\frac{2\pi j_0}{n_k} - t\right)\right|\left|b_k\left(\frac{2\pi j_0}{n_k} - t\right) + 1\right| \\ &\qquad + \left|B_{k-1}\left(\frac{2\pi j_0}{n_k} - t\right) - B_{k-1}\left(\frac{2\pi j_0}{n_k}\right)\right| \\ &\qquad < 2^{-k} + 2^{-k} = 2^{-k+1} \end{aligned} \tag{2.3.13}$$

and

$$\begin{aligned} &\left|B_k\left(\frac{2\pi j_0}{n_k} + \frac{\pi}{n_k} - t\right) - B_{k-1}\left(\frac{2\pi j_0}{n_k}\right)\right| \\ &\qquad \le \left|B_{k-1}\left(\frac{2\pi j_0}{n_k} + \frac{\pi}{n_k} - t\right)\right|\left|b_k\left(\frac{2\pi j_0}{n_k} + \frac{\pi}{n_k} - t\right) - 1\right| \\ &\qquad + \left|B_{k-1}\left(\frac{2\pi j_0}{n_k} + \frac{\pi}{n_k} - t\right) - B_{k-1}\left(\frac{2\pi j_0}{n_k}\right)\right| \\ &\qquad < 2^{-k} + 2^{-k} = 2^{-k+1}. \end{aligned} \tag{2.3.14}$$

On the other hand, using (2.2.3), (2.3.11) and (2.3.13), we get

$$\begin{aligned}
&\left|\Phi_{r'}(x+\lambda(r'),B_k)+B_{k-1}\left(\frac{2\pi j_0}{n_k}\right)\right| \\
&\qquad =\left|\int_{\mathbb{T}}\varphi_{r'}(t)B_k(x+\lambda(r')-t)dt+B_{k-1}\left(\frac{2\pi j_0}{n_k}\right)\right| \\
&\qquad =\left|\int_{\mathbb{T}}\varphi_{r'}(t)\left[B_k\left(\frac{2\pi j_0}{n_k}-t\right)+B_{k-1}\left(\frac{2\pi j_0}{n_k}\right)\right]dt\right| \\
&\qquad \le\left|\int_{I}\varphi_{r'}(t)\left[B_k\left(\frac{2\pi j_0}{n_k}-t\right)+B_{k-1}\left(\frac{2\pi j_0}{n_k}\right)\right]dt\right| \\
&\qquad +\left|\int_{I^c}\varphi_{r'}(t)\left[B_k\left(\frac{2\pi j_0}{n_k}-t\right)+B_{k-1}\left(\frac{2\pi j_0}{n_k}\right)\right]dt\right| \\
&\qquad \le 2^{-k+1}\int_I\varphi_{r'}(t)dt+2\cdot 2^{-k}\le 4\cdot 2^{-k}.
\end{aligned} \tag{2.3.15}$$

Similarly, using (2.3.14), we conclude

$$\left|\Phi_{r''}\left(x+\lambda(r''),B_k\right)-B_{k-1}\left(\frac{2\pi j_0}{n_k}\right)\right|\le 4\cdot 2^{-k}. \tag{2.3.16}$$

From (2.3.12), (2.3.15) and (2.3.16) it follows that

$$\left|\Phi_{r'}(x+\lambda(r'),B)-\Phi_{r''}\left(x+\lambda(r''),B\right)\right|\ge 1-16\cdot 2^{-k},$$

which implies the divergence of $\Phi_r(x+\lambda(r),B)$ at a point x. The theorem is proved. □

CHAPTER 3

Differentiation bases in $\mathbb{R}^n$

3.1 Introduction

This chapter is devoted to differentiation bases in $\mathbb{R}^n$, which is defined as follows:

Definition 3.1. *A family $\mathcal{B}$ of bounded, positively measured sets from $\mathbb{R}^n$ is said to be a differentiation basis (or simply basis), if for any point $x \in \mathbb{R}^n$ there exists a sequence of sets $E_k \in \mathcal{B}$ such that $x \in E_k$, $k = 1, 2, \ldots$ and* $\operatorname{diam}(E_k) \to 0$ *as $k \to \infty$.*

We remind the classical theorems determining the optimal Orlicz space for the functions having almost everywhere differentiable integrals with respect to the bases of rectangles $\mathcal{R}^n$.

Theorem I (Jessen–Marcinkiewicz–Zygmund, [15])**.** $L(1+\log^+ L)^{n-1}(\mathbb{R}^n) \subset \mathcal{F}(\mathcal{R}^n)$.

Theorem J (Saks, [35])**.** *If the convex function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies*

$$\Psi(t) = o(t \log^{n-1} t) \text{ as } t \to \infty,$$

then $\Psi(L)(\mathbb{R}^n) \not\subset \mathcal{F}(\mathcal{R}^n)$. Moreover, there exists a positive function $f \in \Psi(L)(\mathbb{R}^n)$ such that $\delta_{\mathcal{R}^n}(x, f) = \infty$ everywhere.

The optimal Orlicz space remains the same if we consider the basis $\mathcal{DR}^n$ instead of $\mathcal{R}^n$. The first part follows from the embedding $L(1+\log^+ L)^{n-1}(\mathbb{R}^n) \subset \mathcal{F}(\mathcal{R}^n) \subset \mathcal{F}(\mathcal{DR}^n)$ and the second can be deduced from the following

Theorem K (Zerekidze, [42] (see also [43, 44]))**.** $\mathcal{F}^+(\mathcal{DR}^n) = \mathcal{F}^+(\mathcal{R}^n)$.

However, the set of functions having almost everywhere differentiable integrals with respect to these bases can differ. In Section 3.3 we prove that the condition $\gamma_\Delta < \infty$ is necessary and sufficient for the full equivalency of rare dyadic basis $\mathcal{DR}^2_\Delta$ and complete dyadic bases $\mathcal{DR}^n$.

Theorem 3.1. *If $\Delta = \{\nu_k\}$ is an increasing sequence of positive integers with*

$$\gamma_\Delta = \sup_{k\in\mathbb{N}}(\nu_{k+1} - \nu_k) < \infty,$$

then

$$\mathcal{F}(\mathcal{DR}^2_\Delta) = \mathcal{F}(\mathcal{DR}^2).$$

Theorem 3.2. *If $\Delta = \{\nu_k\}$ is an increasing sequence of positive integers with $\gamma_\Delta = \infty$, then there exists a function $f \in \mathcal{F}(\mathcal{DR}^2_\Delta)$ such that*

$$\limsup_{\mathrm{len}(R)\to 0,\, x\in R\in\mathcal{DR}^2} \left| \frac{1}{|R|}\int_R f(t)\,dt \right| = \infty$$

for any $x \in \mathbb{R}^n$.

In Section 3.4 we prove that two quasi-equivalent subbases of some density basis differentiate the same class of non-negative functions. In Section 3.5 we apply this theorem for bases formed of rectangles.

Definition 3.2. *A basis $\mathcal{B}$ is said to be density basis if $\mathcal{B}$ differentiates the integral of any characteristic function $\mathbb{I}_E$ of measurable set E:*

$$\delta_{\mathcal{B}}(x, \mathbb{I}_E) = 0 \textit{ at almost every } x \in \mathbb{R}^n.$$

We will say that the basis $\mathcal{B}$ differentiates a class of functions $\mathcal{F}$, if basis $\mathcal{B}$ differentiates the integrals of all functions of $\mathcal{F}$.

Definition 3.3. *Let $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{B}$ be subbases. We will say that basis $\mathcal{B}_2$ is quasi-coverable by basis $\mathcal{B}_1$ (with respect to basis $\mathcal{B}$) if for any $R \in \mathcal{B}_2$ there exist $R_k \in \mathcal{B}_1$, $k = 1, 2, \ldots, p$ and*

$R' \in \mathcal{B}$ such that

$$R \subseteq \tilde{R} \subseteq R', \quad \tilde{R} = \bigcup_{k=1}^{p} R_k \tag{3.1.1}$$

$$\operatorname{diam}(R') \le c \cdot \operatorname{diam}(R), \quad |R'| \le c|R_k|, \quad k = 1, 2, \ldots, p, \tag{3.1.2}$$

$$\sum_{k=1}^{p} |R_k| \le c|\tilde{R}|, \quad |\tilde{R}| \le c|R|, \tag{3.1.3}$$

where constant $c \ge 1$ depends only on bases $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}$. We will say two bases are quasi-equivalent if they are quasi-coverable with respect to each other.

Theorem 3.3. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be subbases of density basis $\mathcal{B}$ formed of open sets from $\mathbb{R}^n$. If the bases $\mathcal{B}_1$ and $\mathcal{B}_2$ are quasi-equivalent with respect to $\mathcal{B}$ then*

$$\mathcal{F}^+(\mathcal{B}_1) = \mathcal{F}^+(\mathcal{B}_2).$$

3.2 Some definitions and auxiliary lemmas

Denote by $\overline{E}$ and $\mathring{E}$ the closure and the interior of a set $E \subset \mathbb{R}^2$ respectively, $\mathbb{I}_E$ denotes the indicator function of E. For a given rectangle $R \in \mathcal{R}^2$ we denote by $\operatorname{len}(R)$ the length of the bigger side of R. A set $E \subset \mathbb{R}^2$ is said to be simple, if it can be written as a union of squares of the form

$$\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right) \times \left[\frac{j-1}{2^n}, \frac{j}{2^n}\right), \quad i, j, n \in \mathbb{Z}.$$

If n is the minimal integer with this relation, then we write $\operatorname{wd}(E) = 2^{-n}$. Note that if E is a dyadic rectangle, then $\operatorname{wd}(E)$ coincides with the length of the smaller side of E. If E is a square, then $\operatorname{len}(E) = \operatorname{wd}(E)$. Denote

$$E_{ij}(n) = \bigcup_{k=0}^{n-1} \left[\frac{i}{2}, \frac{i}{2} + \frac{1}{2^{k+1}}\right) \times \left[\frac{j}{2}, \frac{j}{2} + \frac{1}{2^{n-k}}\right), \tag{3.2.1}$$

$$\begin{aligned} F_{ij}(n) &= \left[\frac{i}{2}, \frac{i}{2} + \frac{1}{2^n}\right) \times \left[\frac{j}{2}, \frac{j}{2} + \frac{1}{2^n}\right) \\ &= \bigcap_{k=0}^{n-1} \left[\frac{i}{2}, \frac{i}{2} + \frac{1}{2^{k+1}}\right) \times \left[\frac{j}{2}, \frac{j}{2} + \frac{1}{2^{n-k}}\right) \subset E_{ij}(n), \quad i, j = 0, 1, \end{aligned}$$

and define the sets

$$E(n) = E_{00}(n) \cup E_{01}(n) \cup E_{10}(n) \cup E_{11}(n), \tag{3.2.2}$$

$$F(n) = F_{00}(n) \cup F_{01}(n) \cup F_{10}(n) \cup F_{11}(n) \subset E(n). \tag{3.2.3}$$

Introduce the functions

$$u(x,n) = (n+1)2^{n-2}\left(\mathbb{I}_{F_{00}(n)}(x) + \mathbb{I}_{F_{11}(n)}(x) - \mathbb{I}_{F_{10}(n)}(x) - \mathbb{I}_{F_{01}(n)}(x)\right), \quad n \in \mathbb{N},$$

$$v(x) = \mathbb{I}_{(0,1/2)\times(0,1/2)}(x) + \mathbb{I}_{(1/2,1)\times(1/2,1)}(x) - \mathbb{I}_{(0,1/2)\times(1/2,1)}(x) - \mathbb{I}_{(1/2,1)\times(0,1/2)}(x).$$

Let $\omega \in \mathcal{Q}^2$ be an arbitrary square and ϕ_ω be the linear transformation of $\mathbb{R}^2$ taking ω onto unit square $[0,1)^2 \subset \mathbb{R}^2$. For an arbitrary function $f(x)$ defined on $[0,1)^2$ and for a set $E \subset [0,1)^2$ we define

$$f_\omega(x) = f(\phi_\omega(x)), \quad E_\omega = (\phi_\omega)^{-1}(E) \subset \omega.$$

We have

$$\operatorname{supp}(u_\omega(x,n)) = F_\omega(n), \tag{3.2.4}$$

$$\operatorname{supp}(v_\omega(x)) = \omega, \tag{3.2.5}$$

$$|E_\omega(n)| = \frac{(n+1)|\omega|}{2^n}, \quad |F_\omega(n)| = \frac{|\omega|}{4^{n-1}}, \tag{3.2.6}$$

$$\operatorname{wd}(E_\omega(n)) = \operatorname{wd}(F_\omega(n)) = \operatorname{wd}(\omega)\cdot 2^{-n}. \tag{3.2.7}$$

Simple calculations show that

$$\|u_\omega(x,n)\|_1 = |E_\omega(n)| = \frac{n+1}{2^n}|\omega|, \tag{3.2.8}$$

$$\|v_\omega(x)\|_1 = |\omega|. \tag{3.2.9}$$

Then observe that, if $\omega \in \mathcal{DQ}^2$ is a dyadic square, then for any point $x \in E_\omega(n)$ there exists a dyadic rectangle $R(x) \in \mathcal{DR}^2$ with

$$\frac{1}{|R(x)|}\left|\int_{R(x)} u_\omega(x,n)dx\right| = \frac{n+1}{2}, \quad x \in R(x) \subset E_\omega(n), \tag{3.2.10}$$

$$\operatorname{wd}(R(x)) = \operatorname{wd}(\omega)\cdot 2^{-n}. \tag{3.2.11}$$

Moreover, the rectangle $R(x)$ coincides with $(\phi_\omega)^{-1}$-image of one of the representation rectangles from (3.2.1). Similarly, if $\omega \in \mathcal{DQ}^2$, then

$$\frac{1}{|R(x)|}\left|\int_{R(x)} v_\omega(x)dx\right| = 1, \quad x \in R(x) \subset \omega, \tag{3.2.12}$$

$$\mathrm{wd}(R(x)) = \frac{\mathrm{wd}(\omega)}{2}. \tag{3.2.13}$$

for some square $R(x)$ with $|R(x)| = |\omega|/4$. In this case $R(x)$ coincides with one of the four squares forming ω.

The following simple lemma has been proved in [**?**, 17].

Lemma 3.1. *Let $Q \in \mathcal{DQ}^2$ be an arbitrary dyadic square, $f(x) = f(x_1, x_2) \in L^1(\mathbb{R}^2)$ be a function with* $\operatorname{supp} f(x) \subset Q$ *and*

$$\int_{\mathbb{R}} f(x_1, t)\,dt = \int_{\mathbb{R}} f(t, x_2)\,dt = 0, \quad x_1, x_2 \in \mathbb{R}. \tag{3.2.14}$$

Then for any dyadic rectangle $R \in \mathcal{DR}^2$ satisfying $\mathring{R} \not\subset Q$ we have

$$\int_R f(x)\,dx = 0. \tag{3.2.15}$$

Proof. We suppose

$$Q = [\alpha_1, \beta_1) \times [\alpha_2, \beta_2), \quad R = [a_1, b_1) \times [a_2, b_2).$$

If $R \cap Q = \varnothing$, then (3.2.15) is trivial. Otherwise we will have either $[\alpha_1, \beta_1) \subset [a_1, b_1)$ or $[\alpha_2, \beta_2) \subset [a_2, b_2)$. In the first case, using (3.2.14), we get

$$\begin{aligned}\int_R f(x)\,dx &= \int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x_1, x_2)\,dx_1\,dx_2 \\ &= \int_{a_2}^{b_2}\int_{\alpha_1}^{\beta_1} f(x_1, x_2)\,dx_1\,dx_2 \\ &= \int_{a_2}^{b_2}\left(\int_{\mathbb{R}} f(x_1, x_2)\,dx_1\right)dx_2 = 0.\end{aligned}$$

The second case is proved similarly. □

Lemma 3.2. *Let m be a positive integer and Q be a dyadic square. Then for any simple set $E \subsetneq [0,1)^2$, there exists a finite family Ω of dyadic squares $\omega \subset Q$ such that*

$$E_\omega \cap E_{\omega'} = \varnothing, \quad \omega \neq \omega', \tag{3.2.16}$$

$$\min_{\omega\in\Omega} \operatorname{wd}(\omega) = \operatorname{wd}(Q) \cdot (\operatorname{wd}(E))^m, \tag{3.2.17}$$

$$\left| Q \setminus \bigcup_{\omega\in\Omega} E_\omega \right| = |Q| \left(1 - |E|\right)^m. \tag{3.2.18}$$

Proof. Define a sequence of sets G_k, $k = 1, 2, \ldots, m$, with

$$Q = G_1 \supset G_2 \supset \ldots \supset G_m, \tag{3.2.19}$$

and finite families of dyadic squares $\Omega_k \subset \mathcal{DQ}^2$, $k = 1, 2, \ldots, m+1$, such that

$$\operatorname{wd}(\omega) = \operatorname{wd}(Q) \cdot (\operatorname{wd}(E))^{k-1}, \quad \omega \in \Omega_k, \quad k = 1, 2, \ldots, m+1, \tag{3.2.20}$$

$$G_k = \bigcup_{\omega\in\Omega_k} \omega, \quad k = 1, 2, \ldots, m+1, \tag{3.2.21}$$

$$G_k = G_{k-1} \setminus \bigcup_{\omega\in\Omega_{k-1}} E_\omega = \bigcup_{\omega\in\Omega_{k-1}} \left(\omega \setminus E_\omega\right), \quad k = 2, \ldots, m+1. \tag{3.2.22}$$

We do it by induction. For the first step of induction we take just $G_1 = Q$ and let Ω_1 consist of a single rectangle Q. Suppose we have already chosen the sets G_k and the families Ω_k for $k = 1, 2, \ldots, p$, satisfying (3.2.19)$-$(3.2.22). Set

$$G_{p+1} = G_p \setminus \bigcup_{\omega\in\Omega_p} E_\omega = \bigcup_{\omega\in\Omega_p} \left(\omega \setminus E_\omega\right).$$

From the induction hypothesis of (3.2.20) it follows that

$$\operatorname{wd}\left(\omega \setminus E_\omega\right) = \operatorname{wd}(\omega) \cdot \operatorname{wd}(E) = \operatorname{wd}(Q) \cdot (\operatorname{wd}(E))^p.$$

Hence we conclude that G_{p+1} is a union of dyadic squares with side lengths $\operatorname{wd}(Q)\cdot(\operatorname{wd}(E))^p$ and we define the family Ω_{p+1} as a collection of these squares. Thus we get G_{p+1} and Ω_{p+1} satisfying the conditions (3.2.19)$-$(3.2.22) for $k = p+1$, that completes the induction process. Applying (3.2.8), (3.2.21) and (3.2.22) we obtain

$$|G_k| = |G_{k-1}| - \left| \bigcup_{\omega\in\Omega_{k-1}} E_\omega \right| = |G_{k-1}| - |E||G_{k-1}| = (1 - |E|)\,|G_{k-1}|$$

and therefore

$$|G_{m+1}| = (1-|E|)^m\,|Q|. \tag{3.2.23}$$

Obviously the family of squares $\Omega = \cup_{k=1}^{m+1}\Omega_k$ satisfies the hypothesis of the lemma. Indeed, suppose $\omega, \omega' \in \Omega$ are arbitrary squares. If $\omega, \omega' \in \Omega_k$ for some k, then according to (3.2.20) we have $\omega \cap \omega' = \varnothing$ and so (3.2.16). If $\omega \in \Omega_k$, $\omega' \in \Omega_{k'}$ and $k < k'$, then

$$E_{\omega'} \subset \omega' \subset G_{k'},$$

$$E_\omega \subset G_k \setminus G_{k+1} \Rightarrow E_\omega \cap G_{k'} = \varnothing.$$

Thus we again get (3.2.16). The condition (3.2.17) immediately follows from (3.2.20), and (3.2.18) follows from (3.2.23) and from the relation

$$\left|\bigcup_{\omega\in\Omega} E_\omega\right| = \left|\bigcup_{k=1}^{m+1}\bigcup_{\omega\in\Omega_k} E_\omega\right| = \left|\bigcup_{k=1}^{m+1} G_k \setminus G_{k+1}\right| = |Q\setminus G_{m+1}| = |Q|(1-(1-|E|)^m).$$

□

Lemma 3.3. *Let $L > 1$ be a positive integer and $Q \in \mathcal{DQ}^2$ be a dyadic square. Then there exist a function $f \in L^\infty(\mathbb{R}^2)$, numbers $\alpha(L) \in \mathbb{N}$ and $\beta(L) > 0$, depended on L, such that*

$$\operatorname{supp} f \subset Q, \tag{3.2.24}$$

$$\|f\|_\infty \le \beta(L), \tag{3.2.25}$$

$$|\operatorname{supp} f| \le \frac{2|Q|}{\beta(L)}, \tag{3.2.26}$$

$$\operatorname{wd}(\operatorname{supp} f) \ge \operatorname{wd}(Q)\cdot 2^{-\alpha(L)}, \tag{3.2.27}$$

$$\int_R f(x)dx = 0, \quad R \in \mathcal{DR}^2, \quad \mathring{R} \not\subset Q, \tag{3.2.28}$$

and for any point $x \in Q$ there exists a rectangle $R(x) \subset Q$ satisfying

$$\operatorname{wd}(R(x)) \ge \operatorname{wd}(Q)\cdot 2^{-\alpha(L)}, \tag{3.2.29}$$

$$\frac{1}{|R(x)|}\left|\int_{R(x)} f(t)dt\right| \ge L. \tag{3.2.30}$$

Proof. Let $n = 2L$ and denote

$$\alpha(L) = n(2^n + 1), \quad \beta(L) = (n+1)2^{n-2}, \tag{3.2.31}$$

$$m = m(L) = \left[\frac{2^n(\ln(n+1) + (n-2)\ln 2)}{n+1}\right] + 1 < 2^n. \tag{3.2.32}$$

Let $E = E(n)$ be the set defined in (3.2.2). We have $|E(n)| = (n+1)/2^n$ and $\mathrm{wd}(E(n)) = 2^{-n}$. Applying Lemma 3.2, we may find family Ω of dyadic squares $\omega \subset Q$ with properties (3.2.16)–(3.2.18). Set

$$G = \bigcup_{\omega\in\Omega} E_\omega(n), \quad G_1 = Q \setminus G. \tag{3.2.33}$$

According to (3.2.18), (3.2.31) and (3.2.32), we have

$$|G_1| = (1 - |E(n)|)^m |Q| = \left(1 - \frac{n+1}{2^n}\right)^m |Q| < \frac{|Q|}{\beta(L)}$$

From (3.2.17) and (3.2.32) it follows that

$$G_1 = \bigcup_{\omega\in\Omega_1} \omega,$$

where Ω_1 is a family of squares with

$$\min_{\omega\in\Omega_1} \mathrm{wd}(\omega) = \min_{\omega\in\Omega} \mathrm{wd}(\omega) = \mathrm{wd}(Q)\cdot(\mathrm{wd}(E(n)))^m \geq \mathrm{wd}(Q)\cdot 2^{-n\cdot 2^n}. \tag{3.2.34}$$

Define

$$f(x) = \sum_{\omega\in\Omega} u_\omega(x,n) + \beta(L)\sum_{\omega\in\Omega_1} v_\omega(x) = g(x) + g_1(x).$$

Clearly this function satisfies (3.2.24) and (3.2.25). Then, we have

$$\operatorname{supp} g = \bigcup_{\omega\in\Omega} F_\omega(n) \subset G, \quad \operatorname{supp} g_1 = G_1,$$

$$\operatorname{supp} f = \operatorname{supp} g \bigcup \operatorname{supp} g_1.$$

This together with (3.2.6) and (3.2.33) implies

$$\begin{aligned}|\operatorname{supp} f| &= \bigcup_{\omega\in\Omega} |F_\omega(n)| + |G_1| \\ &= \frac{1}{(n+1)2^{n-2}}\sum_{\omega\in\Omega} |E_\omega(n)| + |G_1| \\ &= \frac{1}{(n+1)2^{n-2}}|G| + |G_1| \leq \frac{2|Q|}{\beta(L)}\end{aligned}$$

and therefore we get (3.2.26). Using (3.2.34), we obtain

$$\operatorname{wd}(\operatorname{supp} g) \geq \min_{\omega\in\Omega} \operatorname{wd}(\omega)\cdot \operatorname{wd}(F(n)) = \operatorname{wd}(Q)\cdot 2^{-n(2^n+1)} = \operatorname{wd}(Q)\cdot 2^{-\alpha(L)},$$

$$\operatorname{wd}(\operatorname{supp} g_1) \geq \min_{\omega\in\Omega_1} \operatorname{wd}(\omega) \geq \operatorname{wd}(Q)\cdot 2^{-n\cdot 2^n} > \operatorname{wd}(Q)\cdot 2^{-\alpha(L)},$$

and therefore we get (3.2.27). The condition (3.2.28) follows from Lemma 3.1, since $f(x)$ satisfies the condition (3.2.14) according the definitions of functions $u_\omega(x,n)$ and $v_\omega(x)$. To prove (3.2.30) we take an arbitrary point $x \in Q$. We have either $x \in G$ or $x \in G_1$. In the first case we will have $x \in E_\omega(n)$ for some square $\omega \in \Omega$. By (3.2.10) there exists a dyadic rectangle $R = R(x)$, $x \in R \subset E_\omega(n)$, such that

$$\frac{1}{|R|}\left|\int_R f(t)dt\right| = \frac{1}{|R|}\left|\int_R u_\omega(t,n)dt\right| = \frac{n+1}{2} > L.$$

In the second case from (3.2.12) we obtain

$$\frac{1}{|R|}\left|\int_R f(t)dt\right| = \frac{\beta(L)}{|R|}\left|\int_R v_\omega(t)dt\right| \geq 2^n > L$$

for some square $R = R(x)$, $x \in R \subset \omega$. Obviously in any case $R(x)$ satisfies (3.2.29). Lemma is proved. □

3.3 Dyadic rectangles in $\mathbb{R}^2$

Proof of Theorem 3.1. Let $\Delta = \{\nu_k\}$ be a sequence with $\gamma_\Delta < \infty$. Suppose conversely, we have

$$\mathcal{F}(\mathcal{DR}^2_\Delta) \setminus \mathcal{F}(\mathcal{DR}^2) \neq \varnothing.$$

That means there exist a function $f \in L_{\text{loc}}(\mathbb{R}^2)$, a number $\alpha > 0$ and a set $E \subset \mathbb{R}^n$ with $|E| > 0$ such that

$$\delta_{\mathcal{DR}^2_\Delta}(x,f) = 0, \quad \text{a.e.,} \tag{3.3.1}$$

$$\delta_{\mathcal{DR}^2}(x,f) > \alpha, \quad x \in E, \tag{3.3.2}$$

According to (3.3.1) for almost any $x \in \mathbb{R}^2$ one can choose a number $\delta(x) > 0$ such that the conditions

$$x \in R \in \mathcal{DR}^2_\Delta, \quad \operatorname{len}(R) < \delta(x),$$

imply

$$\left| \frac{1}{|R|} \int_R f - f(x) \right| < \frac{\alpha}{2}. \tag{3.3.3}$$

For some $\delta > 0$ the set $F = \{x \in E \colon \delta(x) \geq \delta\} \subset E$ has positive measure. Then, using the representation

$$F = \bigcup_{j \in \mathbb{Z}} \left\{ x \in F \colon \frac{j\alpha}{2} \leq f(x) < \frac{(j+1)\alpha}{2} \right\},$$

we find a set

$$G = \left\{ x \in F \colon \frac{j_0\alpha}{2} \leq f(x) < \frac{(j_0+1)\alpha}{2} \right\} \subset F \tag{3.3.4}$$

having positive measure. Combining (3.3.2), (3.3.3) and (3.3.4), we will have

$$\delta_{\mathcal{DR}^2}(x, f) > \alpha, \quad x \in G, \tag{3.3.5}$$

$$\left| \frac{1}{|R|} \int_R f - f(x) \right| < \frac{\alpha}{2}, \text{ if } x \in R \cap G,\ R \in \mathcal{DR}^2(\Delta),\ \operatorname{len}(R) < \delta, \tag{3.3.6}$$

$$\sup_{x,y \in G} |f(x) - f(y)| \leq \frac{\alpha}{2}. \tag{3.3.7}$$

Since almost all points of G are density points, we may fix $x_0 \in G$ with

$$\lim_{\operatorname{len}(R) \to 0,\, x_0 \in R \in \mathcal{DR}^2} \frac{|R \cap G|}{|R|} = 1.$$

Using this relation and (3.3.5), we find a rectangle

$$R' = \left[\frac{p-1}{2^n}, \frac{p}{2^n} \right) \times \left[\frac{q-1}{2^m}, \frac{q}{2^m} \right),$$

such that

$$x_0 \in R' \in \mathcal{DR}^2, \quad \operatorname{len}(R') < \delta, \tag{3.3.8}$$

$$\left| \frac{1}{|R'|} \int_{R'} f - f(x_0) \right| > \alpha, \tag{3.3.9}$$

$$|R' \cap G| > (1 - 4^{-\gamma_\Delta})|R'|, \tag{3.3.10}$$

Besides, we may suppose

$$\nu_{k_t-1} < n \le \nu_{k_t}, \quad \nu_{k_s-1} < m \le \nu_{k_s}, \tag{3.3.11}$$

for some integers t and s. This and the definition of γ_Δ imply that R' is a union of rectangles of the form

$$\left[\frac{i-1}{2^{\nu_{k_t}}}, \frac{i}{2^{\nu_{k_t}}}\right) \times \left[\frac{j-1}{2^{\nu_{k_s}}}, \frac{j}{2^{\nu_{k_s}}}\right) \in \mathcal{DR}^2_\Delta,$$

and from (3.3.9) it follows that at least for one of these rectangles, say R'', we have

$$\left|\frac{1}{|R''|}\int_{R''} f - f(x_0)\right| > \alpha. \tag{3.3.12}$$

From the definition of γ_Δ and (3.3.11) we get

$$|R''| = \frac{1}{2^{\nu_{k_t}+\nu_{k_s}}} \ge \frac{1}{2^{\nu_{k_t}+\nu_{k_s}-\nu_{k_t-1}-\nu_{k_s-1}}} \cdot \frac{1}{2^{n+m}} \ge |R'| \cdot 4^{-\gamma_\Delta}.$$

From this and (3.3.10) we obtain $R'' \cap G \ne \varnothing$. Take a point $x_1 \in R'' \cap G$. From (3.3.7) and (3.3.12) we get

$$\left|\frac{1}{|R''|}\int_{R''} f - f(x_1)\right| > \left|\frac{1}{|R''|}\int_{R''} f - f(x_0)\right| - |f(x_1) - f(x_0)| > \frac{\alpha}{2}. \tag{3.3.13}$$

On the other hand we have $x_1 \in R'' \cap G$, $R'' \in \mathcal{DR}^2_\Delta$, $\operatorname{len}(R'') \le \operatorname{len}(R') < \delta_0$, and therefore by (3.3.6) we obtain

$$\left|\frac{1}{|R''|}\int_{R''} f - f(x_1)\right| < \alpha/2.$$

The last relation together with (3.3.13) gives a contradiction, which completes the proof of the theorem. □

Proof of Theorem 3.2. Now we suppose $\gamma_\Delta = \infty$, which means there exists a sequence of integers $p_k \nearrow \infty$ such that

$$\lim_{k\to\infty} (\nu_{p_k+1} - \nu_{p_k}) = \infty. \tag{3.3.14}$$

Using this relation, we may find sequences of integers L_k and l_k, $k = 1, 2, \ldots$, such that

$$l_{k+1} > l_k + \alpha(L_k), \quad k = 1, 2, \ldots, \tag{3.3.15}$$

$$\nu_{p_k} < l_k < l_k + \alpha(L_k) < \nu_{p_k+1}, \quad k = 1, 2, \ldots, \tag{3.3.16}$$

$$L_{k+1} > 2^k \cdot (\beta(L_k) + k) \quad k = 1, 2, \ldots, \tag{3.3.17}$$

where $\alpha(L)$ and $\beta(L)$ are the constants taken from Lemma 3.3. Applying Lemma 3.3 for the numbers $L = L_k$, $l = l_k$ and for the square

$$Q = Q_{ij}^k = \left[\frac{i-1}{2^{l_k}}, \frac{i}{2^{l_k}}\right) \times \left[\frac{j-1}{2^{l_k}}, \frac{j}{2^{l_k}}\right), \quad 1 \le i, j \le 2^{l_k},$$

we get functions $f_{ij}^k \in L^\infty(\mathbb{R}^2)$ satisfying the conditions

$$\operatorname{supp} f_{ij}^k \subset Q_{ij}^k, \tag{3.3.18}$$

$$\|f_{ij}^k\|_\infty \le \beta(L_k), \tag{3.3.19}$$

$$|\operatorname{supp} f_{ij}^k| \le \frac{2|Q_{ij}^k|}{\beta(L_k)}, \tag{3.3.20}$$

$$\operatorname{wd}(\operatorname{supp} f_{ij}^k) \ge 2^{-l_k-\alpha(L_k)}, \tag{3.3.21}$$

$$\int_R f_{ij}^k(x)dx = 0, \quad R \in \mathcal{DR}^2, \quad \mathring{R} \not\subset Q_{ij}^k, \tag{3.3.22}$$

and for any point $x \in Q_{ij}^k$ there exists a dyadic rectangle $R_k(x) \subset Q_{ij}^k$ with

$$\operatorname{wd}(R_k(x)) \ge 2^{-l_k-\alpha(L_k)}, \tag{3.3.23}$$

$$\frac{1}{|R_k(x)|}\left|\int_{R_k(x)} f_{ij}^k(t)dt\right| \ge L_k. \tag{3.3.24}$$

Define the function

$$F_k(x) = \sum_{i,j=1}^{2^{l_k}} f_{ij}^k(x).$$

From the relations (3.3.18)–(3.3.24) we conclude

$$|\operatorname{supp} F_k| \le \frac{2}{\beta(L_k)}, \tag{3.3.25}$$

$$\operatorname{wd}(\operatorname{supp} F_k) \ge 2^{-l_k-\alpha(L_k)}, \tag{3.3.26}$$

$$\|F_k\|_\infty \le \beta(L_k), \tag{3.3.27}$$

$$\int_R F_k(x)dx = 0, \quad R \in \mathcal{DR}^2, \quad \operatorname{len}(R) \ge 2^{-l_k}, \tag{3.3.28}$$

and for any point $x \in [0,1)^2$ there exists a dyadic rectangle $R_k(x) \subset [0,1)^2$ such that

$$2^{-l_k} > \operatorname{len}(R_k(x)) \ge \operatorname{wd}(R_k(x)) \ge 2^{-l_k-\alpha(L_k)}, \tag{3.3.29}$$

$$\frac{1}{|R_k(x)|}\left|\int_{R_k(x)} F_k(t)dt\right| \ge L_k. \tag{3.3.30}$$

Denote

$$F(x)=\sum_{k=1}^{\infty}\frac{F_k(x)}{2^k}. \tag{3.3.31}$$

From (3.3.25) and (3.3.16) it follows that $\|F_k\|_1\le 2$ and so $\|F\|_1\le 2$. Let $x\in[0,1)^2$ be an arbitrary point. From the relations (3.3.15) and (3.3.29) we get $\mathrm{len}(R_k(x))\ge 2^{-l_{k+1}}\ge 2^{-l_j}$ if $j>k$. Thus, using (3.3.28), we obtain

$$\int_{R_k(x)}F_j(t)dt=0,\quad j>k. \tag{3.3.32}$$

On the other hand the relations (3.3.27) and (3.3.17) imply

$$\left|\frac{1}{|R_k(x)|}\int_{R_k(x)}\sum_{j=1}^{k-1}\frac{F_j(t)}{2^j}dt\right|\le\beta(L_{k-1})<\frac{L_k}{2},\quad k\ge 2. \tag{3.3.33}$$

From (3.3.30), (3.3.32) and (3.3.33) we get the inequality

$$\left|\frac{1}{|R_k(x)|}\int_{R_k(x)}F(t)dt\right|\ge\frac{1}{|R_k(x)|}\left|\int_{R_k(x)}F_k(t)dt\right|-\frac{L_k}{2}>\frac{L_k}{2},$$

which yields

$$\limsup_{\mathrm{len}(R)\to 0,\,x\in R\in\mathcal{DR}^2}\left|\frac{1}{|R|}\int_R F(t)dt\right|=\infty,\quad x\in[0,1)^2. \tag{3.3.34}$$

Now take an arbitrary rectangle $R\in\mathcal{DR}^2_\Delta$. We have

$$\mathrm{len}(R)=2^{-\nu_k}\ge\mathrm{wd}(R)=2^{-\nu_t}. \tag{3.3.35}$$

From (3.3.28) we get

$$\int_R F_j(t)dt=0\ \text{ if } l_j\ge\nu_k. \tag{3.3.36}$$

On the other hand if $l_j<\nu_k$, then from (3.3.16) it follows that

$$l_j+\alpha(L_j)<\nu_k$$

and therefore by (3.3.26) we get

$$\mathrm{wd}(\mathrm{supp}\,(F_j))\ge 2^{-l_j-\alpha(L_j)}\ge 2^{-\nu_k}. \tag{3.3.37}$$

Thus, using simple properties of dyadic rectangles, we conclude that

$$l_j<\nu_k,\ R\not\subset\mathrm{supp}\,(F_j)\Rightarrow R\cap\mathrm{supp}\,(F_j)=\varnothing. \tag{3.3.38}$$

Consider the sets

$$G_1 = \{x \in [0,1)^2 : \delta_{\mathcal{R}}(x, F_k) = 0,\ k = 1, 2, \ldots\},$$
$$G_2 = \bigcup_{k=1}^{\infty} \bigcap_{j:\, l_j \geq \nu_k}^{\infty} \Big([0,1)^2 \setminus \operatorname{supp}(F_j)\Big),$$
$$G = G_1 \cap G_2.$$

Since $F_k(x)$ is bounded, the equality $\delta_{\mathcal{R}}(x, F_k) = 0$ holds almost everywhere and so $|G_1| = 1$. From (3.3.25) it follows that $|G_2| = 1$ and therefore we get $|G| = 1$. Take an arbitrary point $x \in G$. We have

$$x \notin \operatorname{supp}(F_j), \quad j > k_0, \tag{3.3.39}$$

for some k_0. Consider the rectangle $R \in \mathcal{DR}^2_\Delta$ such that $x \in R$. Suppose we have (3.3.35) and $k > k_0$. Then form (3.3.38) and (3.3.39) we get

$$R \cap \operatorname{supp}(F_j) = \varnothing, \text{ if } j > k_0 \text{ and } l_j < \nu_k. \tag{3.3.40}$$

From (3.3.36) and (3.3.40) we conclude

$$\frac{1}{|R|} \int_R F(t)dt = \sum_{j=1}^{k_0} \frac{1}{2^j \cdot |R|} \int_R F_j(t)dt.$$

Thus we obtain

$$\lim_{\operatorname{len}(R) \to 0,\, x \in R \in \mathcal{DR}^2_\Delta} \frac{1}{|R|} \int_R F(t)dt = \sum_{j=1}^{k_0} \frac{F_j(x)}{2^j}. \tag{3.3.41}$$

On the other hand (3.3.39) implies

$$F(x) = \sum_{j=1}^{k_0} \frac{F_j(x)}{2^j}. \tag{3.3.42}$$

From (3.3.41) and (3.3.42) we conclude that $F \in \mathcal{F}(\mathcal{DR}^2_\Delta)$ and $\operatorname{supp} F \subset [0,1)^2$. To have a function f defined on entire $\mathbb{R}^2$ we set

$$f(x) = f(x_1, x_2) = F(\{x_1\}, \{x_2\}), \quad x \in \mathbb{R}^2.$$

Clearly $f \in \mathcal{F}(\mathcal{DR}^2_\Delta)$ and (3.3.34) holds for any $x \in \mathbb{R}^2$. □

3.4 Quasi-equivalent bases in $\mathbb{R}^n$

Proof of Theorem 3.3. First, let us suppose that

$$\mathcal{F}^+(\mathcal{B}_1) \setminus \mathcal{F}^+(\mathcal{B}_2) \neq \varnothing.$$

That means there exists a non-negative function $f \in L_{\text{loc}}(\mathbb{R}^n)$ such that

$$\delta_{\mathcal{B}_1}(x, f) = 0, \quad \text{a.e.}, \tag{3.4.1}$$

$$\delta_{\mathcal{B}_2}(x, f) > 0, \quad x \in E_1, \tag{3.4.2}$$

where $|E_1| > 0$. From (3.4.2) it follows that there exist such positive numbers α and γ that the set

$$E_2 = \{x \in \mathbb{R}^n \colon \delta_{\mathcal{B}_2}(x, f) > \alpha,\, 0 \leq f(x) \leq \gamma\} \tag{3.4.3}$$

has positive measure. Set $f = f_\gamma + f^\gamma$, where

$$f_\gamma(x) = \begin{cases} f(x), & \text{if } 0 \leq f(x) \leq \gamma, \\ 0, & \text{if } f(x) > \gamma. \end{cases}$$

Since $\mathcal{B}_2 \subseteq \mathcal{B}$ is density basis, then it differentiates L^∞ and therefore differentiates $f_\gamma \in L^\infty$, namely we have $\delta_{\mathcal{B}_2}(x, f_\gamma) = 0$ almost everywhere. Denote by E_3 the subset of E_2 where $\delta_{\mathcal{B}_2}(x, f_\gamma) = 0$. Clearly $|E_3| = |E_2| > 0$. From this we can deduce that if $x \in E_3 \subset E_2$ then $\delta_{\mathcal{B}_2}(x, f) = \delta_{\mathcal{B}_2}(x, f^\gamma)$ and $f^\gamma(x) = 0$, since $0 \leq f(x) \leq \gamma$. Furthermore, using (3.4.1), we get set $E_4 \subset E_3$ of positive measure such that for any $x \in E_4$

$$\delta_{\mathcal{B}_2}(x, f^\gamma) > \alpha, \quad f^\gamma(x) = 0, \tag{3.4.4}$$

$$\delta_{\mathcal{B}_1}(x, f^\gamma) = 0, \tag{3.4.5}$$

According to (3.4.5) for any $x \in E_4$ one can choose a number $\delta(x) > 0$ such that the conditions

$$x \in R \in \mathcal{B}_1, \quad \text{diam}(R) < \delta(x),$$

imply

$$\frac{1}{|R|}\int_R f^\gamma(u)\,du < \eta,$$

where $\eta > 0$ will be conveniently chosen later. For some $\delta > 0$ the set $G = \{x \in E_4 \colon \delta(x) \geq \delta\}$ has positive measure. Thus, we have transformed (3.4.4) and (3.4.5) into

$$\delta_{\mathcal{B}_2}(x, f^\gamma) > \alpha,\ f^\gamma(x) = 0, \quad \text{if } x \in G, \tag{3.4.6}$$

$$\frac{1}{|R|}\int_R f^\gamma(u)\,du < \eta,\ \text{if } R \cap G \neq \varnothing,\ R \in \mathcal{B}_1,\ \operatorname{diam}(R) < \delta. \tag{3.4.7}$$

Since $\mathcal{B}$ differentiates $\mathbb{I}_G$, hence we may fix $x_0 \in G$ with

$$\lim_{\operatorname{diam}(R)\to 0,\, x_0 \in R \in \mathcal{B}} \frac{|R \cap G|}{|R|} = 1,$$

which means that for any $\varepsilon > 0$ there exists $\sigma(\varepsilon)$ such that $\operatorname{diam}(R) < \sigma(\varepsilon)$ and $x_0 \in R \in \mathcal{B}$ imply $|R \cap G| > (1-\varepsilon)|R|$. Using this relation and (3.4.6), we can fix such R that

$$x_0 \in R \in \mathcal{B}_2, \quad \operatorname{diam}(R) < \frac{1}{c}\min\left(\sigma\left(\frac{1}{c}\right), \delta\right), \tag{3.4.8}$$

$$\frac{1}{|R|}\int_R f^\gamma(u)\,du > \alpha, \tag{3.4.9}$$

As we have that basis $\mathcal{B}_2$ is quasi-coverable with $\mathcal{B}_1$, then for $R \in \mathcal{B}_2$ we can fix $R' \in \mathcal{B}$ and $R_k \in \mathcal{B}_1$, $k = 1, 2, \ldots, p$ such that (3.1.1),(3.1.2) and (3.1.3) hold. From this and (3.4.8) we get

$$x_0 \in R' \in \mathcal{B}, \quad \operatorname{diam}(R') < \sigma\left(\frac{1}{c}\right),$$

which implies

$$|R' \cap G| > \left(1 - \frac{1}{c}\right)|R'|. \tag{3.4.10}$$

which together with (3.1.2) gives that there exists $x_k \in R_k \cap G$, $k = 1, 2, \ldots, p$. Now, since each R_k contains some point from G, we can use (3.4.7) and come to contradiction against (3.4.9). Namely, combining (3.4.7), (3.4.8) and (3.1.2) we have $x_k \in R_k \cap G$, $R_k \in \mathcal{B}_1$, $\operatorname{diam}(R_k) < \delta$ and therefore

$$\frac{1}{|R_k|}\int_{R_k} f^\gamma(u)\,du < \eta, \quad k = 1, 2, \ldots, m$$

which together with (3.1.3) implies

$$\int_{\tilde{R}} f^\gamma(u)\,du \le \int_{\tilde{R}} \sum_{k=1}^{p} \mathbb{I}_{R_k}(u) f^\gamma(u)\,du$$
$$= \sum_{k=1}^{p} \int_{R_k} f^\gamma(u)\,du < \eta \sum_k |R_k| \le \eta c |\tilde{R}|$$

and

$$\frac{1}{|\tilde{R}|} \int_{\tilde{R}} f^\gamma(u)\,du < \eta c.$$

On the other hand, from non-negativity of function f^γ and from (3.4.9),(3.1.3) it follows

$$\frac{1}{|\tilde{R}|} \int_{\tilde{R}} f^\gamma(u)\,du \ge \frac{|R|}{|\tilde{R}|} \cdot \frac{1}{|R|} \int_{R} f^\gamma(u)\,du > \frac{\alpha}{c},$$

which is impossible if choose $\eta < \frac{\alpha}{c^2}$. Thus we have proved that $\mathcal{F}^+(\mathcal{B}_1) \subset \mathcal{F}^+(\mathcal{B}_2)$.

In the same way we can prove the inverse inclusion $\mathcal{F}^+(\mathcal{B}_2) \subset \mathcal{F}^+(\mathcal{B}_1)$. Therefore the theorem is proved. □

3.5 Applications

In this section we give several corollaries from Theorem 3.3 for bases formed of rectangles. First of all, notice that if we change the sets of some basis $\mathcal{B}$ by arbitrary sets of measure zero, then we get a new basis $\tilde{\mathcal{B}}$ with the same differentiation properties as $\mathcal{B}$. In particular, $\delta_{\mathcal{B}}(x, f) = \delta_{\tilde{\mathcal{B}}}(x, f)$ for any $x \in \mathbb{R}^n$ and $f \in L_{\text{loc}}(\mathbb{R}^n)$. The reason for this is that we use Lebesgue integral, which is consistent if we modify the domain of integration by a set of measure zero. Hence, we can extend Theorem 3.3 for bases formed of bounded sets, which are open up to a set of measure zero.

It is well known that the basis of all rectangles $\mathcal{R}^n$ differentiates $L^\infty(\mathbb{R}^n)$, i.e. it is a density basis. Therefore we can apply Theorem 3.3 for $\mathcal{B} = \mathcal{R}^n$ and get a criteria for two bases formed of rectangles differentiating the same class of non-negative functions:

Corollary 3.1. *If bases $\mathcal{R}_1$ and $\mathcal{R}_2$ formed of rectangles in $\mathbb{R}^n$ are quasi-equivalent, then $\mathcal{F}^+(\mathcal{R}_1) = \mathcal{F}^+(\mathcal{R}_2)$.*

Let $\Omega = \{\omega_k^i\}_{i,k=1}^{n,\infty}$ be finite family of sequences with

$$\omega_k^i \to 0 \text{ as } k \to \infty \text{ for } i = 1, 2, \ldots, n. \tag{3.5.1}$$

Define the basis $\mathcal{R}_\Omega$ as a family of rectangles of the form

$$\prod_{i=1}^{n} \left[(m_i - 1)\omega_{k_i}^i, m_i\omega_{k_i}^i\right), \quad m_i \in \mathbb{Z},\, k_i \in \mathbb{N},\, i = 1, 2, \ldots, n.$$

and the basis $\tilde{\mathcal{R}}_\Omega$ as a family of rectangles with side lengths $l_i, i = 1, 2, \ldots, n$ satisfying

$$c_1 \cdot \omega_{k_i}^i \le l_i \le c_2 \cdot \omega_{k_i}^i, \quad k_i \in \mathbb{N},\, i = 1, 2, \ldots, n.$$

Then, it can be shown that the bases $\mathcal{R}_\Omega$ and $\tilde{\mathcal{R}}_\Omega$ are quasi-equivalent. Therefore

Corollary 3.2. *For any Ω with (3.5.1)*

$$\mathcal{F}^+(\tilde{\mathcal{R}}_\Omega) = \mathcal{F}^+(\mathcal{R}_\Omega).$$

Corollary 3.3. *If the family of sequences Ω satisfies*

$$\max_{1\le i\le n} \sup_{k\in\mathbb{N}} \frac{\omega_k^i}{\omega_{k+1}^i} < \infty, \tag{3.5.2}$$

then

$$\mathcal{F}^+(\mathcal{R}_\Omega) = \mathcal{F}^+(\mathcal{R}^n). \tag{3.5.3}$$

Proof. Denote by γ the finite quantity of the left hand side of (3.5.2). Then for coefficients $c_1 = 1$ and $c_2 = \gamma + 1$ we have $\mathcal{F}^+(\tilde{\mathcal{R}}_\Omega) = \mathcal{F}^+(\mathcal{R}^n)$. Hence from the theorem we deduce (3.5.3). □

Finally, if we take $\omega_k^i = 2^{-\nu_k}$, $k \in \mathbb{N}$, $i = 1, 2, \ldots, n$, where $\Delta = \{\nu_k : k \ge 1\}$ is an increasing sequence of positive integers, the basis $\mathcal{R}_\Omega$ becomes the basis of all dyadic rectangles $\mathcal{DR}_\Delta^n$ corresponding to the sequence Δ.

Corollary 3.4. *If the sequence $\Delta = \{\nu_k\}$ satisfies $\gamma_\Delta < \infty$, then*

$$\mathcal{F}^+(\mathcal{DR}_\Delta^n) = \mathcal{F}^+(\mathcal{R}^n).$$

Particularly, if we take $\Delta = \mathbb{N}$, we get Theorem K.

Conclusion

The thesis comprises three chapters.

In Chapter 1, it is investigated generalizations of the theorem of Fatou for convolution type integral operators with general approximate identities. It is introduced $\lambda(r)-$convergence, which is a generalization of non-tangential convergence in the unit disc. The connections between general approximate identities and optimal convergence regions for such operators are described in different functional spaces.

1. It is found a necessary and sufficient condition on $\lambda(r)$ that ensures almost everywhere $\lambda(r)-$convergence for convolution type integral operators in both spaces of bounded measures and integrable functions. Moreover, in the case of bounded measures, the convergence occurs at any point where the measure is differentiable. In the case of integrable functions, the convergence occurs at any Lebesgue point of the function.

2. It is discovered a necessary and sufficient condition on $\lambda(r)$ that provides almost everywhere $\lambda(r)-$convergence for the same convolution type integral operators in the space of essentially bounded functions. Additionally, the convergence occurs at any Lebesgue point of the function.

In Chapter 2, it is studied some generalizations of the theorem of Littlewood, which makes an important complement to the theorem of Fatou, constructing analytic function possesing almost everywhere divergent property along a given tangential curve. The same convolution type integral operators are considered with more general kernels than approximate identities. Two kinds of generalizations of the theorem of Littlewood are obtained possessing everywhere divergent property.

3. Under general assumptions, it is constructed a characteristic function such that the

convolution with general kernels possesses everywhere divergent property along a given tangential curve. Particularly, it is proved that there exists a bounded harmonic function having everywhere strong divergent property along a given tangential curve.

4. Under general assumptions, it is constructed a bounded function, which is the boundary values of some Blaschke product, such that the convolution with general kernels owns everywhere divergent property along a given tangential curve.

Chapter 3 is devoted to some questions of equivalency of differentiation bases in $\mathbb{R}^n$. The full equivalence of basis of rare dyadic rectangles and the basis of complete dyadic rectangles in $\mathbb{R}^2$ is investigated. It is introduced quasi-equivalence between two differentiation bases in $\mathbb{R}^n$ and is considered the set of functions that such bases differentiate.

5. It is found a necessary and sufficient condition for the full equivalence of basis of rare dyadic rectangles and the basis of complete dyadic rectangles in $\mathbb{R}^2$.

6. It is proved that two quasi-equivalent bases of some density basis in $\mathbb{R}^n$ differentiate the same set of non-negative functions.

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