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Karen Alexandr Keryan

Convergence, uniqueness and basis problems for orthonormal spline systems

SYNOPSIS

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Պաշտոնական ընդդիմախոսներ՝	Ֆիզ. մաթ. գիտ. դոկտոր Փ. Ֆ. Ք. Մյուլլեր Ֆիզ. մաթ. գիտ. դոկտոր ՈՒ. Գոգինավա ՀՀ ԳԱԱ թղթակից անդամ, Ֆիզ. մաթ. գիտ. դոկտոր Ա.Ա. Սահակյան
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Official referees:	Dr. Habil. P. F. X. Müller Dr. of Phys. Math. Sciences U. Goginava Corresponding member of NAS of RA, Dr. of Phys. Math. Sciences A.A. Sahakyan
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Scientific secretary of specialized council
Dr. of Phys. Math. Sciences T.N. Harutyunyan

General Characterization of the Work

<u>**Actuality of the work.**</u> In the thesis the convergence, uniqueness and basis problems for various orthonormal spline systems are studied. The Franklin system is one of the simplest orthonormal spline system. The Franklin system was constructed by Ph. Franklin in 1928 ([32]) as the first example of an orthonormal basis for $C[0,1]$. It is a complete orthonormal system of continuous, piecewise linear functions (with dyadic knots). The Franklin system is obtained by applying the Gram-Schmidt orthogonalization process to the Faber-Schauder system.

The systematic investigations of the Franklin system have been started by Z. Ciesielski with his remarkable papers [10] and [11]. Since then, the Franklin system has been studied by many authors from different points of view. The basic properties of this system, including exponential estimates for the Franklin functions and L^p-stability on dyadic blocks, have been obtained by Z. Ciesielski in [10] and [11]. These properties turned out to be an important tool in further investigations of the Franklin system. It is known that this system is a basis in $C[0,1]$ and L^p for $1 \le p < \infty$. The unconditionality of the Franklin system in $L^p, 1 < p < \infty$, has been proved by S. V. Bochkarev in [6]. Moreover, the Franklin system is an unconditional basis in all reflexive Orlicz spaces ([5]). The existence of an unconditional basis in H^1 has been first proved by B. Maurey [66], but the proof was non-constructive. The first explicit construction of an unconditional basis in H^1 is due to L. Carleson [7]. Then, P. Wojtaszczyk has obtained a characterization of the BMO space in terms of the coefficients of a function in the Franklin system and proved that the Franklin system is an unconditional basis in the real Hardy space H^1 ([89]). The unconditionality of the Franklin system in real Hardy spaces $H^p, 1/2 < p \le 1$, has been obtained by P. Sjölin and J. Strömberg ([84]). Z. Ciesielski and Sun-Yung A. Chang proved that $f \in H^1$ iff its Fourier-Franklin series is unconditionally convergent in L^1 (cf. [8]).

The Franklin system has had important applications in various problems of analysis. In particular, the constructions of bases in spaces $C^1(I^2)$ (see [12], [78]) and $A(D)$ (see [5]) are based on this system. Here $C^1(I^2)$ is the space of all continuously-differentiable functions $f(x,y)$ on the square $I^2 = [0,1] \times [0,1]$ with the norm

$$\|f\| = \max|f(x,y)| + \max\left|\frac{\partial f}{\partial x}\right| + \max\left|\frac{\partial f}{\partial y}\right|,$$

and $A(D)$ denotes the space of analytic functions on the open disc $D = \{z : |z| < 1\}$ that are continuously extendable up to the boundary. The norm of a function $f \in A(D)$ is defined by

$$\|f\| = \max_{|z|\le 1}|f(z)|.$$

The questions of existence of bases in $C^1(I^2)$ and $A(D)$ were posed by S. Banach [4].

Further investigations showed that Haar and Franklin systems share a lot of properties. Nevertheless the proofs of properties of Franklin system essentially differ from that of Haar system. In particular, a Cantor type uniqueness theorem for Franklin series was obtained quite recently ([52]). Note that there are a lot of properties of the Haar system, that are neither proved nor disproved for the Franklin system. For example it is not known how one can reconstruct the coefficients of an everywhere convergent Franklin series. The analogous question for the Haar system was studied back in 1960s by V.A. Skvorcov [85].

The extensions of results of the Franklin system to orthonormal spline systems with arbitrary knots first have been considered in the case of piecewise linear systems, i.e. general Franklin systems (orthonormal spline systems of order 2). This was possible due to precise estimates of the inverse to the Gram matrix of piecewise linear B-spline bases with arbitrary knots. The general Franklin systems were introduced by Z. Ciesielski and A. Kamont [16].

Franklin system can be generalized by changing
a. the sequence of knots;
b. the smoothness of functions;
c. the domain of definition, e.g. $[0,1]$ can be replaced with R, or a polygon $P \subset R^2$, etc.

In the thesis generalizations of the Franklin system in above mentioned three directions or their combinations are considered. We investigate the basis properties of those systems, the convergence and uniqueness problems, as well as the relationship between the majorant and the Paley square function.

The aim and objectives of the thesis. The main aim of the present thesis is to study the convergence, uniqueness and basis problems for orthonormal spline systems. The following results are obtained:

1. Description of all sequences for which the corresponding periodic Franklin system is a basis or an unconditional basis in the Lebesgue or Hardy spaces, or in the space $B^1(T)$.
2. Equivalence of the L^p norms over dyadic intervals of Paley square function and the majorant of partial sums of series in the classical Franklin system under some condition.
3. Description of all sequences for which the corresponding orthogonal spline system is an unconditional basis in the Hardy space $H^1[0,1]$.
4. Description of all sequences for which the corresponding orthonormal spline system with zero means is a basis in the Hardy space $H^1(\mathbb{R})$.
5. Description of all sequences for which the corresponding Franklin system with zero means is an unconditional basis in the Hardy space $H^1(\mathbb{R})$.
6. Equivalence of absolutely and unconditionally almost everywhere convergence of series in general Franklin systems.
7. Construction of systems of piecewise linear functions such that the Fourier series in that system of any continuous function with a prescribed growth rate is locally uniformly convergent.
8. Coefficient reconstruction formulas are obtained for almost everywhere convergent Haar, Walsh, Franklin and Strömberg series with a regular majorant of the partial sums.

The methods of the investigation. In the thesis the methods of metric theory of functions and functional analysis are used.

Scientific novelty. All of the main results are new. Let us emphasize some of the results. We give a simple geometric characterization of knot sequences for which the corresponding orthonormal spline system of arbitrary order k is an unconditional basis in the Hardy space $H^1[0,1]$. We then find characterizations of all the sequences of knots for which the orthonormal spline system of order r with zero mean on $\mathbb{R}$ forms a basis in $H^1(\mathbb{R})$. In the case $r=2$ we obtain characterizations of sequences for which the corresponding system is an unconditional basis in $H^1(\mathbb{R})$. These characterizations are not direct analogues of those obtained for the segment $[0,1]$. New type of uniqueness theorems are obtained for series in the Haar, Walsh and Franklin systems.

Theoretical and practical value. The basis problem of a system in a Banach space is one of the central problems in functional analysis and metric theory of functions. Let us mention a few results included in the thesis. It is obtained the characterization of all sequences for which the corresponding orthogonal spline system or Franklin system with zero means on $\mathbb{R}$ is an unconditional basis in the corresponding real Hardy space. We also characterize all the sequences for which the corresponding orthonormal spline system with zero means on $\mathbb{R}$ is a basis in $H^1(\mathbb{R})$. Another important problem in functional analysis is the uniqueness of Fourier series. Several uniqueness results are also obtained. It is proved that if f is a sum of almost everywhere (a.e.) convergent Haar, Walsh or Franklin series with a regular majorant of the partial sums, then the coefficients of that series can be reconstructed from truncations of the function f.

Approbation of the results. The main results of the thesis have been presented at the international conferences [23*-29*] and at the seminars of the chair of function theory and mathematical analysis at YSU, at the seminars of Academician G.G. Gevorkyan, at the Approximation Theory Seminar, Gdansk University/Polish Academy of Sciences, IM PAN Gdansk, Poland, at the Functional Analysis Seminar, Polish Academy of Sciences, IM PAN Warsaw, Poland and at the Analysis Seminar, Johannes Kepler University, Linz, Austria.
Publications. The main results of the thesis have been published in 22 scientific articles. The list of the articles is given at the end of the Synopsis.
The structure and the volume of the thesis. The thesis consists of introduction, 4 chapters and a list of references. The number of references is 115. The volume of the thesis is 208 pages.

The Main Content of the Thesis

In Introduction we recall several classical results concerning the Franklin system obtained by S. V. Bochkarev, P. Wojtaszczyk, P. Sjölin and J. Strömberg, Z. Ciesielski and Sun-Yung A. Chang. We also state two important applications of the Franklin system in construction of bases in the space of all continuously-differentiable functions on the unit square and in the space of analytic functions on the open unit disc that are continuously extendable up to the boundary. The questions of existence of bases in these spaces were posed by S. Banach. Further we give the definition of the general Franklin system and indicate three possible generalizations of the classical Franklin system. In the thesis generalizations of the Franklin system in that three directions or their combinations are be considered.

In Chapter 1 we study the basis and unconditional basis properties of periodic Franklin system in Lebesgue, Hardy spaces and in $B^1(T)$. The relationship between the integrability of Paley square function P and the majorant of partial sums S^* with respect to classical Franklin system $\{\mathbf{f}_n\}_{n=0}^{\infty}$ is also studied. In Section 1.3 we introduce orthogonal spline systems of any fixed order and characterize all sequences for which the corresponding orthogonal spline system forms a basis for the real Hardy space $H^1[0,1]$. Now let us represent these and other results in more detail.

Let us begin with the results concerning Franklin systems corresponding to arbitrary grid points. The first results were obtained in [16] and [45]. G.G. Gevorkyan and A. Kamont [45] proved that the Franklin systems are unconditional bases in $L^p[0,1]$ under some conditions on the generating sequence. Moreover, the spaces BMO and $\mathrm{Lip}(\alpha)$, for $0 < \alpha < 1$ were characterized in terms of Franklin coefficients in [45]. Further G.G. Gevorkyan and A. Sahakyan [47] proved the unconditionality of Franklin systems in $L^p[0,1]$ under weaker conditions. The final result concerning the unconditionality of Franklin systems in $L^p[0,1]$ was obtained by G.G. Gevorkyan and A. Kamont.

Theorem 1.1.A([48]) Let $\{f_n(t)\}_{n=0}^{\infty}$ be the Franklin system corresponding to an admissible sequence $\mathcal{T}$. Then $\{f_n(t)\}_{n=0}^{\infty}$ forms an unconditional basis in each $L^p[0,1]$, $1 < p < \infty$.

For a UMD space X, the problem of unconditionality of the classical Franklin system in $L^p(X)$ has been solved by T. Figiel, who developed a general martingale approach to such problems in [29] and [30]. Summaries of Figiel's work appear in [31] and [68].

A. Kamont and P. F. X. Müller [60] extended Theorem 1.1.4 to UMD valued spaces. They proved the unconditionality of the general Franklin systems in $L^p(X)$ under some conditions on the structure and regularity of the sequence of knots, where X is a UMD space.

G.G. Gevorkyan and A. Kamont gave a simple geometric characterization of knot sequences for which the corresponding general Franklin system is a basis or an unconditional basis in $H^1[0,1]$, cf.

[49]. We note that in both papers [48] and [49], an essential tool for their proof is the association of a so called characteristic interval to each Franklin function f_n.

Let us give the definition of periodic Franklin system corresponding to an arbitrary sequence.

Denote by T the periodic interval $[0,1)$. Intervals in T are $[a,b)$, with $0 \leq a < b \leq 1$ or with $a > b$, $a, b \in T$. In the latter case the interval $[a,b)$ denotes $[a,1) \cup [0,b)$. The length of $I = [a,b)$ is denoted by $|I|$; and in the case $a > b$ the length of the interval is the sum of the lengths of $[a,1)$, and $[0,b)$.

Definition 1.1.1. A sequence $\mathcal{T} = \{t_n : n \geq 0\} \subset [0;1)$ is called admissible on T if $\mathcal{T}$ is dense in $T = [0;1)$, $t_0 = 0$, $t_n \in T$, $n \geq 1$, and $t_i \neq t_j$ for $i \neq j$.

We say that a function f is continuous on T if it is continuous at any $x \in [0,1)$ and $f(0) = f(1-)$ where $f(1-) = \lim_{x\to 1-} f(x)$.

Let $\mathcal{T} = \{t_n : n \geq 0\}$ be an admissible sequence on T. For $n \geq 2$, let $\mathcal{T}_n$ be the ordered sequence of points consisting of the grid points $(t_j)_{j=0}^{n-1}$, i.e., $\mathcal{T}_n$ is of the form

$$\mathcal{T}_n = \{\tau_{n,i} : 0 = \tau_{n,0} < \tau_{n,1} < \ldots < \tau_{n,n-2} < \tau_{n,n-1} < 1\}.$$

For any n denote $\tau_{n,i+nk} = \tau_{n,i}$, for $i = 0,1,...,n-1$, $k \in \mathbb{Z}$. Let S_n be the space of continuous on T functions defined on $[0;1)$, linear on each interval $(\tau_{n,i};\tau_{n,i+1}), 0 \leq i \leq n-1$. Clearly $\dim S_n = n$ and $S_{n-1} \subset S_n$. Hence for $n \geq 2$ there exists a unique function (up to sign) $\mathfrak{f}$ which belongs to S_n, is orthogonal to S_{n-1} and $\|\mathfrak{f}_n\|_2 = 1$. This function is called *the nth periodic Franklin function corresponding to the sequence* $\mathcal{T}$.

Definition 1.1.2. The periodic Franklin system $\{\mathfrak{f}_n(x) : n \geq 1\}$ corresponding to a sequence $\mathcal{T}$ is defined as follows; $\mathfrak{f}_1(x) \equiv 1$, and for $n \geq 2$ the function $\mathfrak{f}_n(x)$ is the nth Franklin function corresponding to the sequence $\mathcal{T}$.

For any n denote by $\lambda_{n,i}$ the length of the segment $[\tau_{n,i-1};\tau_{n,i}]$, i.e. $\lambda_{n,i} = |[\tau_{n,i-1};\tau_{n,i}]|$, $i \in \mathbb{Z}$.

Definition 1.1.3. Let $k \in \mathbb{N}$. An admissible sequence $\mathcal{T}$ is called *k-regular on T with a parameter γ,* if

$$\frac{1}{\gamma} \leq \frac{\lambda_{n,i+1} + \lambda_{n,i+2} + \ldots + \lambda_{n,i+k}}{\lambda_{n,i} + \lambda_{n,i+1} + \ldots + \lambda_{n,i+k-1}} \leq \gamma, \quad \text{for} \quad n \geq 2,\ 0 \leq i \leq n-1.$$

Let us emphasize that for a 1-regular sequence on T the quotient of the lengths of the neighbouring intervals $[0,\tau_{n,1})$ and $[\tau_{n,n-1},1)$ of the point 0 is uniformly bounded and uniformly bounded away from 0.

Note that Definition 1.1.3 corresponds to the periodic case. Let us also give the definition of regularities in the non-periodic case. Let $t_0 := 0$, $t_1 := 1$ and $\mathcal{T} = (t_n)_{n=2}^{\infty}$ be a dense sequence of points in the open unit interval such that each point occurs at most k times. Such point sequences are called *k-admissible.* For $n \geq 2$, let $\mathcal{T}_n$ be the ordered sequence of points consisting of the grid points $(t_j)_{j=0}^{n}$ counting multiplicities, where the knots 0 and 1 have multiplicity k, i.e., $\mathcal{T}_n$ is of the form

$$\mathcal{T}_n = (0 = \tau_{n,1} = \cdots = \tau_{n,k} < \tau_{n,k+1} \leq \cdots \leq \tau_{n,n+k-1} < \tau_{n,n+k} = \cdots = \tau_{n,n+2k-1} = 1).$$

Definition 1.1.4. Let $k \in \mathbb{N}$ and $(t_n)_{n=0}^{\infty}$ be a k-admissible point sequence. Then, this sequence is called *k-regular with a parameter* $\gamma \geq 1$ if

$$\frac{1}{\gamma} \leq \frac{\lambda_{n,i+1} + \lambda_{n,i+2} + \ldots + \lambda_{n,i+k}}{\lambda_{n,i} + \lambda_{n,i+1} + \ldots + \lambda_{n,i+k-1}} \leq \gamma, \quad \text{for} \qquad n \geq 2, \quad 2 \leq i \leq n+k-1,$$

where $\lambda_{n,i} = \tau_{n,i} - \tau_{n,i-1}$.

So, in other words, (t_n) is k-regular if there is a uniform finite bound $\gamma \geq 1$ such that for all n the ratios of the lengths of the neighbouring supports of B-spline functions of order k in the grid $\mathcal{T}_n$ are bounded by γ.

Clearly, if a sequence is k-regular on T in the periodic case, then it will be also k-regular in the non-periodic case, but not vice versa. Let us show an example revealing the mentioned difference.

Consider the following recursively defined sequence. Let $t_0 = 0$. At the first step we add the midpoint of the segment $[0,1]$, i.e. $1/2$. At the second step we add the points $1/8, 2/8, 3/8$, splitting the segment $[0,1/2]$ into four equal parts, and then the midpoint of $[1/2,1]$, i.e. $3/4$. At the nth step we add three points to the leftmost interval, so that they split the interval into 4 equal parts and then we consecutively add from left to right the midpoints of the intervals generated by the points defined before the nth step. In this way we will obtain a sequence, which is non-periodically 1-regular with parameter 4, but not periodically 1-regular on T with any parameter.

Basis and unconditional basis properties of the periodic Franklin systems with arbitrary knots in spaces $L^p(T)$ and $H^1(T)$ are also considered.

Theorem 1.1.5.([1*]) Let $\mathcal{T}$ be an admissible sequence on T and $\{\mathfrak{f}_n(t)\}_{n=1}^{\infty}$ be the corresponding periodic Franklin system. Then the system $\{\mathfrak{f}_n(t)\}_{n=1}^{\infty}$ is an unconditional basis in each $L^p(T),\ 1 < p < \infty$.

Using the estimates for the periodic Franklin functions the author and M. Poghosyan [2*] obtained characterizations of those sequences for which the corresponding periodic Franklin system forms a basis or an unconditional basis in $H^1(T)$. Let us recall the definitions of the real Hardy spaces $H^1[0,1]$ and $H^1(T)$.

Definition 1.1.6. Let Δ be either $[0,1]$ or the periodic interval T. A function $\phi \in L^1[0,1]$ is called a Δ-atom, if either $\phi \equiv 1$ on Δ, or there is an interval $I \subset \Delta$ such that supp $\phi \subset I$, $\sup|\phi| \leq \frac{1}{|I|}$ and $\int_I \phi(t)\mathrm{d}t = 0$.

Note that if ϕ is an atom on $[0,1]$ then it is also an atom on T, but not vice versa. For instance, the function $\phi(t) = 2(\mathbb{1}_{[0,1/4]}(t) - \mathbb{1}_{[3/4,1)}(t))$ is an atom on T, but not on $[0,1]$ (recall $\mathbb{1}_E$ denotes the characteristic function of a set E).

Definition 1.1.7.([19]) Let Δ be either $[0,1]$ or the periodic interval T. A distribution f is said to belong to $H^1(\Delta)$, if there exist Δ-atoms ϕ_i and real coefficients c_i such that $f = \sum_{i=1}^{\infty} c_i\phi_i$, where the convergence is in the sense of distributions. The norm in $H^1(\Delta)$ is defined by $\|f\|_{H^1(\Delta)} = \inf(\sum_{i=1}^{\infty}|c_i|)$, where the infimum is taken over all atomic decompositions of f.

Theorem 1.1.8.([2*]) Let $\mathcal{T}$ be an admissible sequence on T and $\{\mathfrak{f}_n(t)\}_{n=1}^{\infty}$ be the corresponding periodic Franklin system. Then the system $\{\mathfrak{f}_n(t)\}_{n=1}^{\infty}$ is a basis in $H^1(T)$ if and only if the sequence $\mathcal{T}$ is 2-regular on T with some parameter γ.

Theorem 1.1.9.([2*]) Let $\mathcal{T}$ be an admissible sequence on T and $\{\mathfrak{f}_n(t)\}_{n=1}^{\infty}$ be the corresponding periodic Franklin system. Then the system $\{\mathfrak{f}_n(t)\}_{n=1}^{\infty}$ is an unconditional basis in $H^1(T)$ if and only if the sequence $\mathcal{T}$ is 1-regular on T with some parameter γ.

In particular it was given in [2*] a relationship between the subsequent four conditions. Let

(t_n) be an admissible sequence of knots with the corresponding periodic Franklin system $(\mathfrak{f}_n)_{n\geq 1}$. For a sequence $(a_n)_{n\geq 1}$ of coefficients let

$$P := \Big(\sum_{n=1}^{\infty} a_n^2 \mathfrak{f}_n^2\Big)^{1/2} \qquad \text{and} \qquad S^* := \max_{m\geq 1}\Big|\sum_{n=1}^{m} a_n \mathfrak{f}_n\Big|$$

be the Paley square function and the majorant of partial sums, respectively. If $f \in L^1$, then we denote by Pf and S^*f the functions P and S^* corresponding to the coefficient sequence $a_n = \langle f, \mathfrak{f}_n\rangle$, respectively. It was proved in [2*] the equivalence of the following four conditions for any 1-regular sequence on T:

(A) $P \in L^1(T)$

(B) The series $\sum_{n=1}^{\infty} a_n \mathfrak{f}_n$ converges unconditionally in $L^1(T)$,

(C) $S^* \in L^1(T)$,

(D) There exists a function $f \in H^1(T)$ such that $a_n = \langle f, \mathfrak{f}_n\rangle$.

The equivalence of these four conditions implies unconditional basis property of the periodic Franklin system corresponding to any 1-regular sequence on T.

Recall that G.G. Gevorkyan and A. Kamont [49] proved that a Franklin system corresponding to $\mathcal{T}$ is a basis (unconditional basis) in $H^1[0,1]$ if and only if $\mathcal{T}$ is 2-regular (1-regular) with some parameter γ.

Define the space $B^1(T)$ following the work [23].

Definition 1.1.10. A function $b : T \to R$ is called a special atom if either $b(t) \equiv 1$ or $b(t) = \frac{1}{|I|}\mathbb{1}_L(t) - \frac{1}{|I|}\mathbb{1}_R(t)$, where I is an interval on T, L and R are the left and right halves of I respectively.

Definition 1.1.11. The space $B^1(T)$ is defined as follows:

$$B^1(T) = \{f : T \to R; \quad f(t) = \sum_{n=1}^{\infty} c_n b_n(t), \quad \sum_{n=1}^{\infty} |c_n| < \infty\},$$

where b_n is a special atom. Set $\|f\|_{B^1(T)} = \inf \sum_{n=1}^{\infty} |c_n|$, where the infimum is taken over all possible representations of the function f.

The space $B^1[0,1]$ is defined analogously. In this case the supports of the special atoms are ordinary intervals and functions are defined on $[0,1]$.

The space B^1 on the unit circle was defined by G. De Souza in [23], [24].

Clearly, any special atom is also an atom. Hence $B^1(T) \subset H^1(T)$. It is proved in [23] that $B^1(T) \neq H^1(T)$.

The space B^1 on the unit circle coincides with the real parts of the boundary functions of analytic functions g on the unit disc D satisfying the condition (see [25])

$$\int_0^1 \int_0^{2\pi} |g'(re^{i\theta})| \mathrm{d}\theta \mathrm{d}r < \infty.$$

G.G. Gevorkyan [51] recently proved that a Franklin system is a basis or an unconditional basis in $H^1[0,1]$ iff it is a basis or an unconditional basis in $B^1[0,1]$, respectively.

The main result of Section 1.1 is to give a characterization of those knot sequences for which the corresponding periodic Franklin system is a basis or an unconditional basis in $B^1(T)$. The following theorems are proved.

Theorem 1.1.12. Let $\{\mathfrak{f}_n(t)\}_{n=1}^{\infty}$ be the general periodic Franklin system corresponding to an admissible sequence $\mathcal{T}$. Then $\{\mathfrak{f}_n(t)\}_{n=1}^{\infty}$ forms a basis in $B^1(T)$ if and only if the sequence $\mathcal{T}$ is 2-regular on T with some parameter γ.

Theorem 1.1.13. Let $\{\mathfrak{f}_n(t)\}_{n=1}^{\infty}$ be the general periodic Franklin system corresponding to an admissible sequence $\mathcal{T}$. Then $\{\mathfrak{f}_n(t)\}_{n=1}^{\infty}$ forms an unconditional basis in $B^1(T)$ if and only if the sequence $\mathcal{T}$ is 1-regular on T with some parameter γ.

To prove these theorems we essentially use the methods of the paper [51] by G.G. Gevorkyan, together with the properties of periodic Franklin system obtained in [2*], [1*].

To formulate further results obtained in [48], [49], [1*] we need to recall the concept of greedy basis, see S.V. Konyagin, V.N. Temlyakov [63]. Let $(X, \|\cdot\|)$ be a Banach space with a normalized basis $\mathcal{X} = (x_n, n \geq 0)$ (i.e. with $\|x_n\| = 1$). For $x \in X$ and $m \in \mathbb{N}$, let

$$\sigma_m(x) = \inf_{n_1,\ldots,n_m} \inf_{c_1,\ldots,c_m} \|x - \sum_{i=1}^{m} c_i x_{n_i}\|.$$

In addition, for $x = \sum_{n=0}^{\infty} a_n x_n$ and given $m \in \mathbb{N}$, let Λ_m be a subset of indices such that $\operatorname{card}\Lambda_m = m$, (recall that $\operatorname{card} E$ denotes the cardinality of a finite set E) and

$$\min_{n\in\Lambda_m} |a_n| \geq \max_{n\notin\Lambda_m} |a_n|.$$

Denote $G_m(x) = \sum_{n\in\Lambda_m} a_n x_n$. Clearly, $\sigma_m(x) \leq \|x - G_m(x)\|$. Following S.V. Konyagin, V.N. Temlyakov [63], a normalized basis $\mathcal{X} = (x_n, n \geq 0)$ of a Banach space $(X, \|\cdot\|)$ is called *greedy* if there is a constant $C > 0$ such that for all $m \in \mathbb{N}$ and $x \in X$

$$\|x - G_m(x)\| \leq C\sigma_m(x).$$

S.V. Konyagin and V.N. Temlyakov [63] proved that a normalized basis $\mathcal{X} = (x_n, n \geq 0)$ in a Banach space $(X, \|\cdot\|)$ is greedy if and only if it is unconditional and democratic, where democratic means that for any two finite subsets of indices A, B with $\operatorname{card} A = \operatorname{card} B$ the following relation holds:

$$\|\sum_{n\in A} x_n\| \sim \|\sum_{n\in B} x_n\|.$$

G.G. Gevorkyan and A. Kamont [48] proved that the L^p-normalized Franklin system $\{f_{n,p}\}_{n=0}^{\infty}$, where $f_{n,p} = f_n/\|f_n\|_p$, corresponding to an admissible sequence is a greedy basis in $L^p[0,1]$ for $1 < p < \infty$. Moreover, they proved that the H^1-normalized Franklin system $\{f_n/\|f_n\|_{H^1}\}_{n=0}^{\infty}$ corresponding to a 1-regular sequence is a greedy basis in $H^1[0,1]$ (cf. [49]). L^p-normalized periodic Franklin systems corresponding to admissible sequences are also greedy bases in $L^p[0,1]$ for $1 < p < \infty$. (cf. [1*]).

Theorem 1.1.13 implies the following corollary.

Corollary 1.1.14. Let $\mathcal{T}$ be an admissible sequence and $\{\mathfrak{f}_n(t)\}_{n=1}^{\infty}$ be the corresponding general periodic Franklin system. Then the system $\{\mathfrak{f}_n/\|\mathfrak{f}_n\|_{B^1}; n \geq 1\}$ is a greedy basis in $B^1(T)$ if and only if the sequence $\mathcal{T}$ is 1-regular on T with some parameter γ.

In Section 1.2 we are concerned with integrability of Paley square function P and the majorant of partial sums S^* with respect to classical Franklin system. A systematic study of the Franklin system goes back to the papers [10], [11], where, in particular, it was proved that if $f \in L^p[0;1], 1 < p < \infty$, and $\sum_{n=0}^{\infty} a_n \mathbf{f}_n(x)$ is the Fourier-Franklin series of f, then

$$S^*(f,\cdot) \in L^p[0;1], \quad \text{where} \quad S^*(f,x) = \sup_n |S_n(f,x)| \quad \text{and} \quad S_n(f,x) = \sum_{k=0}^{n} a_k \mathbf{f}_k(x). \tag{1}$$

S.V. Bochkarev [6] proved that the Franklin system is an unconditional basis in the space $L^p[0;1]$, $1 < p < \infty$. Moreover, he proved that the Paley operator of the Franklin system is of weak type $(1,1)$, that is, there exists a constant $C > 0$ such that if $f \in L[0;1]$ and $\sum_{n=0}^{\infty} a_n \mathbf{f}_n(x)$ is the Fourier-Franklin series of f, then

$$|\{x \in [0,1] : P(f,x) > \lambda\}| \leq \frac{C}{\lambda} \int_0^1 |f(x)| dx, \tag{2}$$

where $P(f,x) = \left\{\sum_{n=0}^{\infty} a_n^2 \mathbf{f}_n^2(x)\right\}^{1/2}$.

Taking into account that the Paley operator P has $(2,2)$ strong type, that is, $\|P(f,\cdot)\|_2 \leq C\|f\|_2$, by Marcinkiewicz interpolation theorem (see, e.g., [62]), it follows from (2) that for all $p \in (1,\infty)$ we have $\|P(f,\cdot)\|_p \leq C_p\|f\|_p$. Therefore, in view of (1), for any $p > 1$ we have

$$\int_0^1 \sup_n \left|\sum_{k=0}^{n} a_k \mathbf{f}_k(x)\right|^p dx \sim_p \int_0^1 \left(\sum_{k=0}^{\infty} a_k^2 \mathbf{f}_k^2(x)\right)^{p/2} dx. \tag{3}$$

S.-Y. A. Chang and Z. Ciesielski [8] proved that the relation $f \in H^1$ holds iff the square function of the Fourier-Franklin series of f belongs to L^1. It follows from the results by G.G. Gevorkyan [37] and F. Schipp, P. Simon [77] that $f \in H^1$ iff the majorant $S^* \in L^1$. Moreover, G.G. Gevorkyan [37] proved the equivalence of the following conditions for $0 < p \leq 1$:
(A) $P \in L^p$
(B) The series $\sum_{n=1}^{\infty} a_n \mathbf{f}_n$ converges unconditionally in L^p,
(C) $S^* \in L^p$.

Recall that unconditioanlity of the classical Franklin system in Hardy spaces $H^p, 1/2 < p \leq 1$, has been obtained by P. Sjölin and J. Strömberg ([84]); they have also proved that for this range of p, the H^p quasi-norm of $f \in H^p$ is equivalent to the L^p quasi-norm of the square function of the Franklin series with coefficients $a_n = \langle f, \mathbf{f}_n \rangle$. So the conditions (A)-(C) are equivalent to the following condition for $1/2 < p \leq 1$:
(D) There exists a function $f \in H^p$ such that $a_n = \langle f, \mathbf{f}_n \rangle$.

As mentioned above the equivalence of the counterparts of four conditions (A)-(D) in the case $p = 1$ was crucial for the unconditionality in $H^1(T)$ of the periodic Franklin system. Moreover, the Paley function P and the majorant of partial sums S^* have equivalent L^1 norms.

The main results of Section 1.2 concern the relations of $\|P\|_{L^p(I)}$ and $\|S^*\|_{L^p(I)}$ for $p > 0$ and dyadic $I \subset [0,1]$, where P and S^* are Paley function and the majorant of partial sums of series in the classical Franklin system $\{\mathbf{f}_n\}_{n=0}^{\infty}$.

Denote by $L^0(E)$ the metric space of a.e. finite and measurable functions on E, where the metric convergence coincides with the convergence in measure on E.

It was proved in [40] the analogue of relation (3) in the case $p = 0$, as well as its localization on the sets of positive measure. Namely, it was proved in [40] the following result (see Theorems 2.1 - 2.3 in [40]).

Theorem 1.2.A([40]) For a series $\sum_{k=0}^{\infty} a_k \mathbf{f}_k(x)$ the following assertions are equivalent:
1. the series $\sum_{k=0}^{\infty} a_k \mathbf{f}_k(x)$ converges a.e. on E,

2. the series $\sum_{k=0}^{\infty} a_k \mathbf{f}_k(x)$ unconditionally converges in measure on E,
3. $\sup_n |\sum_{k=0}^{n} a_k \mathbf{f}_k(x)| < +\infty$ a.e. on E,
4. $\sum_{k=0}^{\infty} a_k^2 \mathbf{f}_k^2(x) < +\infty$ a.e. on E.

The equivalence of the Franklin system with the Haar system and higher order orthonormal spline systems in L^p and H^p spaces has also beeen studied (see [14], [17], [82], [84]).

In Section 1.2 we show that it is impossible to obtain a localization of the relation (3) even for the dyadic intervals, and we obtain such a localization under some additional conditions. To state the main results of Section 1.2, we introduce some notation. Let $n = 2^\mu + \nu$, where $\mu = 0, 1, 2, ...,$ and $1 \le \nu \le 2^\mu$. Denote

$$s_{n,i} := \begin{cases} \frac{i}{2^{\mu+1}} & \text{for} \quad 0 \le i \le 2\nu, \\ \frac{i-\nu}{2^\mu} & \text{for} \quad 2\nu < i \le n. \end{cases}$$

and $\{n\} := [s_{n,2\nu-2}, s_{n,2\nu}]$. Often the segment $\{n\}$ is called peak interval of the function $\mathbf{f}_n$, because $\mathbf{f}_n$ attains its minimum and maximum values on that interval.

Theorem 1.2.1. For any dyadic segment $I = \left[\frac{æ}{2^k}, \frac{æ+1}{2^k}\right]$, any series $\sum a_n \mathbf{f}_n(x)$ and any number $p > 0$ the following relation holds:

$$\left\| \sup_N \left| \sum_{\{n\} \subset I, n \le N} a_n \mathbf{f}_n \right| \right\|_{L^p(I)} \sim_p \left\| \left\{ \sum_{\{n\} \subset I} a_n^2 \mathbf{f}_n^2 \right\}^{\frac{1}{2}} \right\|_{L^p(I)} .$$

Theorem 1.2.2. For any segment $I = \left[\frac{æ}{2^k}, \frac{æ+1}{2^k}\right] \ne [0,1]$, any $p > 0$ and $C > 0$ there exist series $\sum_{n=0}^{\infty} a_n \mathbf{f}_n(x)$ and $\sum_{n=0}^{\infty} b_n \mathbf{f}_n(x)$ such that

$$\left\| \sup_N \left| \sum_{n \le N} a_n \mathbf{f}_n \right| \right\|_{L^p(I)} > C \cdot \left\| \left\{ \sum_{n=0}^{\infty} a_n^2 \mathbf{f}_n^2 \right\}^{\frac{1}{2}} \right\|_{L^p(I)} , \tag{4}$$

$$\left\| \left\{ \sum_{n=0}^{\infty} b_n^2 \mathbf{f}_n^2 \right\}^{\frac{1}{2}} \right\|_{L^p(I)} > C \cdot \left\| \sup_N \left| \sum_{n \le N} b_n \mathbf{f}_n \right| \right\|_{L^p(I)} . \tag{5}$$

Theorem 1.2.2 shows that the condition $\{n\} \subset I$ in Theorem 1.2.1 is essential.

We show that for the Haar system $\{\chi_n(x)\}$ the analogue of Theorem 1.2.2 is not true, while the analogue of Theorem 1.2.1 is true even without the condition $\{n\} \subset I$. However, in this case the equivalence constants depend also on the segment I. Namely, we have the following result.

Theorem 1.2.3. For any dyadic segment $I = \left[\frac{æ}{2^k}, \frac{æ+1}{2^k}\right]$, any series $\sum a_n \chi_n(x)$ and any number $p > 0$ the following relation holds:

$$\left\| \sup_N \left| \sum_{n \le N} a_n \chi_n \right| \right\|_{L^p(I)} \sim_{p,I} \left\| \left\{ \sum_n a_n^2 \chi_n^2 \right\}^{\frac{1}{2}} \right\|_{L^p(I)} .$$

In Section 1.3 we consider orthonormal spline systems with arbitrary knots, which are generalizations of the Franklin system. The questions of basis property of higher order spline counterparts of Franklin system with arbitrary knots are much more difficult. Note that the problem on uniformly boundedness of the L^∞-norms of orthogonal projections onto spline spaces of order k – was a long-standing problem known as C. de Boor's conjecture (1973) ([21]). The case of $k = 2$ was settled even earlier by Z. Ciesielski [10], the cases $k = 3, 4$ were solved by C. de Boor himself (1968, 1981), [20, 22], but the positive answer in the general case was given by A. Yu. Shadrin [81] in 2001. In order to formulate Shadrin's theorem we need some definitions. Let Π_k be the linear space of all real algebraic polynomials of degree at most k. Let $\Delta : a = t_0 < t_1 < \ldots < t_n = b$ be a finite sequence of knots. The linear space $\mathcal{S}_k(\Delta)$ of polynomial splines of order k with simple knots Δ is defined by

$$\mathcal{S}_k(\Delta) = \{S \in C^{k-2}[a, b] : S|_{[t_i,t_{i+1}]} \in \Pi_{k-1},\ i = 0, 1, \ldots, n-1\}.$$

Let $D[a, b]$ be the linear space of all bounded piecewise continuous real-valued functions on $[a, b]$. The orthogonal projector $\mathcal{P} := \mathcal{P}_k(\Delta) : D[a, b] \to \mathcal{S}_k(\Delta)$ (also called L^2-spline projector) is defined by

$$\|f - \mathcal{P}f\|_2 = \min\{\|f - S\|_2 : S \in \mathcal{S}_k(\Delta)\}, \quad f \in D[a, b].$$

Theorem 1.3.A([81]) For any $k \in \mathbb{N}$, there exists $C > 0$ depending only on k such that the L^2-projector $\mathcal{P} = \mathcal{P}_k(\Delta)$ onto the spline space $\mathcal{S}_k(\Delta)$ satisfies

$$\|\mathcal{P}f\|_\infty \le C\|f\|_\infty, \quad \forall f \in D[a, b].$$

A simplified and shorter proof of this theorem was recently obtained by M. v. Golitschek (2014), cf. [87]. From A.Yu. Shadrin's result it immediately follows that if a sequence of knots is dense in $[0, 1]$, then the corresponding orthonormal spline system of order k is a basis in $L^p[0, 1]$, $1 \le p < \infty$, and $C[0, 1]$.

Let us give the definition of orthonormal spline systems of order k corresponding to arbitrary partitions. Let $k \ge 2$ be an integer and $\mathcal{T} = (t_n)_{n=2}^\infty$ be a k-admissible sequence of points, i.e. $\mathcal{T}$ is dense in the open unit interval and each point occurs at most k times. Moreover, define $t_0 := 0$ and $t_1 := 1$. For n in the range $-k + 2 \le n \le 1$, let $\mathcal{S}_n^{(k)}$ be the space of polynomials of order $n + k - 1$ (or degree $n + k - 2$) on the interval $[0, 1]$ and $(f_n^{(k)})_{n=-k+2}^1$ be the collection of orthonormal polynomials in $L^2 \equiv L^2[0, 1]$ such that the degree of $f_n^{(k)}$ is $n + k - 2$. For $n \ge 2$, let $\mathcal{T}_n$ be the ordered sequence of points consisting of the grid points $(t_j)_{j=0}^n$ counting multiplicities, where the knots 0 and 1 have multiplicity k, i.e., $\mathcal{T}_n$ is of the form

$$\begin{aligned}\mathcal{T}_n = (0 = \tau_{n,1} = \cdots = \tau_{n,k} < \tau_{n,k+1} \le \\ \le \cdots \le \tau_{n,n+k-1} < \tau_{n,n+k} = \cdots = \tau_{n,n+2k-1} = 1).\end{aligned}$$

In that case we also define $\mathcal{S}_n^{(k)}$ to be the space of polynomial splines of order k with grid points $\mathcal{T}_n$. For each $n \ge 2$ the space $\mathcal{S}_{n-1}^{(k)}$ has codimension 1 in $\mathcal{S}_n^{(k)}$ and therefore there exists a function $f_n^{(k)} \in \mathcal{S}_n^{(k)}$ that is orthonormal to the space $\mathcal{S}_{n-1}^{(k)}$. Observe that this function $f_n^{(k)}$ is unique up to sign.

Definition 1.3.1. The system of functions $(f_n^{(k)})_{n=-k+2}^\infty$ is called *orthonormal spline system of order k corresponding to the sequence $(t_n)_{n=0}^\infty$*.

Note that the case $k = 2$ corresponds to orthonormal systems of piecewise linear functions, i.e. to general Franklin systems.

Z. Ciesielski [15] obtained several consequences of Shadrin's result, one of them is an estimate for the inverse of the B-spline Gram matrix. Using this estimate, G.G. Gevorkyan and A. Kamont [50] extended a part of their result from [49] to orthonormal spline systems of arbitrary order and obtained a characterization of knot sequences for which the corresponding orthonormal spline system of order k is a basis in $H^1[0,1]$.

To formulate our result, recall some regularity conditions for a sequence $\mathcal{T}$.

An ℓ-admissible sequence (t_n) is ℓ-regular, if there is a common constant $\gamma \geq 1$ such that for all n the ratios of the lengths of neighbouring supports of B-spline functions of order ℓ in the grid $\mathcal{T}_n$ are bounded by γ (see also Definition 1.1.4).

The following characterization for $(f_n^{(k)})$ to be a basis in H^1 is the main result of [50]:

Theorem 1.3.B. Let $k \geq 1$ and let (t_n) be a k-admissible sequence of knots in $[0,1]$ with the corresponding orthonormal spline system $(f_n^{(k)})$ of order k. Then $(f_n^{(k)})$ is a basis in H^1 if and only if (t_n) is k-regular with some parameter $\gamma \geq 1$.

The further extension requires more precise estimates for the inverse of B-spline Gram matrices of the type known for the piecewise linear case. Such estimates were obtained recently by M. Passenbrunner and A.Yu. Shadrin [74]. Using these estimates, M. Passenbrunner [72] proved the following theorem.

Theorem 1.3.C. Let $k \geq 1$ and let (t_n) be a k-admissible sequence of knots in $[0,1]$ with the corresponding orthonormal spline system $(f_n^{(k)})$ of order k. Then, $(f_n^{(k)})$ is an unconditional basis in $L^p[0,1]$, $1 < p < \infty$, for any k-admissible sequence of knots (t_n).

In Section 1.3 we prove the characterization for $(f_n^{(k)})$ to be an unconditional basis in H^1. The main result of Section 1.3 is the following theorem.

Theorem 1.3.2. Let (t_n) be a k-admissible sequence of points. Then, the corresponding orthonormal spline system $(f_n^{(k)})$ is an unconditional basis in H^1 if and only if (t_n) satisfies the $(k-1)$-regularity condition with some parameter $\gamma \geq 1$.

Note that in the case $k = 2$, i.e. for general Franklin systems, both Theorems 1.3.B and 1.3.2 were obtained by G. G. Gevorkyan and A. Kamont in [49]. (In the terminology of the thesis the condition of strong regularity from [49] is now 1-regularity, and the condition of strong regularity for pairs from [49] is now 2-regularity.)

Corollary 1.3.3. Let $\mathcal{T}$ be a k-admissible sequence and $(f_n^{(k)})$ be the corresponding orthonormal spline system. Then the system $\{f_n^{(k)}/\|f_n^{(k)}\|_{H^1}; n \geq 1\}$ is a greedy basis in $H^1[0,1]$ if and only if the sequence $\mathcal{T}$ is $(k-1)$-regular with some parameter γ.

The case of periodic splines of higher order is not well studied because of arising difficulties. In [3*], [4*] it is proved that L^∞-norms of orthogonal projections onto spaces of periodic splines of order 3 are uniformly bounded. It is possible to extend the Shadrin's theorem to the case of periodic splines and show that there exists a uniform bound for L^∞-norms of orthogonal projections $\widetilde{P}_n$ on spaces of periodic splines of order k with arbitrary order. M. Passenbrunner [73] gave another proof of that fact. He showed that for any integrable function f on the torus, any sequence of its orthogonal projections $(\widetilde{P}_n f)$ onto periodic spline spaces with arbitrary knots $\widetilde{\Delta}_n$ and arbitrary polynomial degree converges to f almost everywhere with respect to the Lebesgue measure, provided the mesh diameter $|\widetilde{\Delta}_n|$ tends to zero.

As we have already mentioned, in the univariate case the L^2-orthogonal projection $\mathcal{P}_V$ onto a spline space V of degree k is bounded as an operator on L^∞ by a constant $C(k)$ depending on the degree k but independent of the knot sequence [81]. P. Oswald [70] proved that the L^2-orthogonal projection $\mathcal{P}_V$ onto spaces $V = V(\mathcal{T})$ of linear splines over triangulations $\mathcal{T}$ of a bounded polygonal domain in $\mathbb{R}^2$ cannot be bounded on L^∞ by a constant that is independent of the underlying triangulation. Similar counterexamples show this for higher dimensions as well. Under different conditions on families of triangulations uniform boundedness of $\|\mathcal{P}_V\|_{L^\infty \to L^\infty}$ was obtained in [71], [88].

Recently A. Seeger and T. Ullrich [80], [79] characterized the range of parameters for which the Haar system is an unconditional basis in the Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}),\ 1 < p, q < \infty$. In [34] it was proved that, for suitable enumerations, the Haar system is a Schauder basis in the classical Sobolev spaces in $\mathbb{R}^d$ with integrability $1 < p < \infty$ and smoothness $1/p - 1 < s < 1/p$.

Another generalization of the Franklin system has been considered by G. Kyriazis, K. Park and P. Petrushev [65]. They considered Franklin systems induced by Courant elements over multilevel nested triangulations of polygonal domains in $\mathbb{R}^2$. They proved that such anisotropic Franklin systems are Schauder bases for C and L^1, and unconditional bases for L^p $(1 < p < \infty)$ and the corresponding Hardy spaces H^1 under some mild conditions imposed on the triangulations. A. Jonsson and A. Kamont [59] discussed the analogues of the Faber-Schauder and Franklin bases for a class of closed sets $F \subset R^n$ admitting a regular sequence of triangulations or generalized triangulations.

In Chapter 2 we introduce orthonormal spline system of a fixed order with zero mean on $\mathbb{R}$, which is another counterpart of the Franklin system. We characterize all the sequences for which the corresponding orthonormal spline system forms a basis for the real Hardy space $H^1(\mathbb{R})$. In Section 2.2 we then find a characterization of all the sequences of knots for which the orthonormal spline system of order 2 with zero mean on $\mathbb{R}$, so called Franklin system with zero means on $\mathbb{R}$ forms an unconditional basis in $H^1(\mathbb{R})$. As we will see these characterizations are not direct analogues of those obtained in [49] for the segment $[0, 1]$.

Let us give the necessary definitions.

Definition 2.1.1. A sequence (partition) $\mathcal{T} = \{t_n : n \geq 0\}$ is called **admissible** if $\mathcal{T}$ is dense in $\mathbb{R}$ and $t_i \neq t_j$, if $i \neq j$.

Let r be a fixed natural number. For an admissible sequence of points $\mathcal{T} = \{t_n, n \geq 0\}$ and $n \geq 1$ we denote by $\mathcal{T}_n$ the ordered sequence of points consisting of the grid points $(t_j)_{j=1}^{n+r+1}$, i.e. $\mathcal{T}_n$ is of the form

$$\mathcal{T}_n = (\tau_{n,1} < \tau_{n,2} < \ldots < \tau_{n,n+r+1}).$$

Denote $\tau_{n,i} := \tau_{n,1}$ for $-r+2 \leq i \leq 0$ and $\tau_{n,i} := \tau_{n,n+r+1}$ for $n+r+2 \leq i \leq n+2r$. Let $(N_{n,i})_{i=-r+2}^{n+r}$ be the sequence of L^∞-normalized B-splines of order r corresponding to $(\tau_{n,i})_{i=-r+2}^{n+2r}$, forming a partition of unity and having the following properties:

$$\operatorname{supp} N_{n,i} = [\tau_{n,i}; \tau_{n,i+r}],\ N_{n,i} \geq 0,\ \int_{\mathbb{R}} N_{n,i}(t)\mathrm{d}t = \frac{\nu_{n,i}}{r},\ \sum_{i=-r+2}^{n+r} N_{n,i}(t) = \mathbb{1}_{\Delta_n}(t),$$

where $\nu_{n,i} := \tau_{n,i+r} - \tau_{n,i}$, $\Delta_n := [\tau_{n,1}; \tau_{n,n+r+1}]$.

Denote by $S_n^{(r)}$ the space of piecewise polynomial functions on $\mathbb{R}$ corresponding to the sequence of knots $\mathcal{T}_n$. That is, $S_n^{(r)}$ is the space of all $r-2$ times continuously differentiable functions on $\mathbb{R}$ with support in $\Delta_n = [\tau_{n,1}; \tau_{n,n+r+1}]$, which are polynomials of degree less than r on each $[\tau_{n,i}; \tau_{n,i+1}]$, $i = 1, 2, ..., n+r$. So $S_n^{(r)}$ is the linear span of the functions $(N_{n,i})_{i=1}^{n+1}$, they form a basis in $S_n^{(r)}$, hence $\dim S_n^{(r)} = n+1$. Notice that $N_{n,i} \notin S_n^{(r)}$ for $i = -r+2, \ldots, 0$ and $i = n+2, \ldots, n+r$.

Definition 2.1.2. Let $S_n^{r,0}$ be the subspace of functions from $S_n^{(r)}$ with zero means on $\mathbb{R}$, i.e.

$$S_n^{r,0} = \left\{ f \in S_n^{(r)}; \int_{\mathbb{R}} f(t)\mathrm{d}t = 0 \right\}.$$

Clearly, $S_{n-1}^{r,0} \subset S_n^{r,0}$ and $S_n^{r,0}$ has codimension 1 in $S_n^{(r)}$, thus $\dim S_n^{r,0} = n$. Therefore, for $n \geq 1$, there exists a unique (up to sign) function $F_n^{(r)} \in S_n^{r,0}$, with the following properties: $\|F_n^{(r)}\|_2 = 1$ and $F_n^{(r)}$ is orthogonal to $S_{n-1}^{r,0}$.

The system of functions $(F_n^{(r)})_{n=1}^{\infty}$ is called *orthonormal spline system with zero means on* $\mathbb{R}$ *of order* r *corresponding to the sequence* $\mathcal{T}$.

The concepts of regularities of partitions $\mathcal{T}$ have played an important role in investigations of both general Franklin systems on $[0; 1]$ and Franklin systems with zero means on $\mathbb{R}$. Now, we introduce the concepts of regularities on $\mathbb{R}$.

Definition 2.1.3. Let $\mathcal{T}$ be an admissible sequence of knots. We say that $\mathcal{T}$ is r**-regular** with a parameter $\gamma \geq 1$, if for any $n \in \mathbb{N}$

$$\gamma^{-1} \leq \frac{\nu_{n,i+1}}{\nu_{n,i}} \leq \gamma, \quad \text{for} \quad i = 1, \ldots, n,$$

where $\nu_{n,i} = \tau_{n,i+r} - \tau_{n,i}$.

Definition 2.1.4. Let $\mathcal{T}$ be an admissible sequence of knots. We say that $\mathcal{T}$ is r**-regular on** $\mathbb{R}$ with a parameter $\gamma \geq 1$, if $\mathcal{T}$ is r-regular and for any $n \in \mathbb{N}$ (recall $\Delta_n = [\tau_{n,1}; \tau_{n,n+r+1}]$)

$$\frac{\nu_{n,1}}{|\Delta_n|} \geq \gamma^{-1}, \quad \frac{\nu_{n,n+1}}{|\Delta_n|} \geq \gamma^{-1}.$$

Let us recall the definition of the real Hardy space $H^1(\mathbb{R})$.

Definition 2.1.5. A function $\phi \in L^1(\mathbb{R})$ is called an atom if there is an interval $I \subset \mathbb{R}$ such that $\operatorname{supp} \phi \subset I$, $\sup |\phi| \leq \frac{1}{|I|}$ and $\int_I \phi(t)\mathrm{d}t = 0$.

Definition 2.1.6.([19]) A distribution f is said to belong to $H^1(\mathbb{R})$, if there exist atoms ϕ_i and real coefficients c_i such that $f = \sum_{i=1}^{\infty} c_i\phi_i$, where the convergence is in the sense of distributions. The norm in $H^1(\mathbb{R})$ is defined as $\|f\|_{H^1} = \inf(\sum_{i=1}^{\infty} |c_i|)$, where the infimum is taken with respect to all atomic decompositions of f.

Clearly, all the functions belonging to $H^1(\mathbb{R})$ have zero mean. Hence, if a system of functions is a basis in $H^1(\mathbb{R})$, then all those functions should have mean zero. That is the reason why we consider orthonormal spline systems with zero means on $\mathbb{R}$.

In Section 2.1 we characterize all the sequences for which the corresponding systems $(F_n^{(r)})_{n=1}^{\infty}$ form a basis in $H^1(\mathbb{R})$. The main result of Section 2.1 is the following:

Theorem 2.1.7. Let $\mathcal{T}$ be an admissible sequence of knots. Then the corresponding orthonormal spline system $(F_n^{(r)})$ with zero means on $\mathbb{R}$ is a basis in $H^1(\mathbb{R})$ if and only if $\mathcal{T}$ is r-regular on $\mathbb{R}$ with some parameter $\gamma \geq 1$.

The particular case of orthonormal spline system with zero means on $\mathbb{R}$ corresponding to $r = 2$ is called *the Franklin system with zero means on* $\mathbb{R}$ and denoted by $(F_n(t))_{n\geq 1}$.

The next result, which is the main result of [13*], characterizes the sequences for which the Franklin system (F_n) forms a basis in the space $H^1(\mathbb{R})$.

Theorem 2.2.1. Let $(F_n)_{n=1}^{\infty}$ be the Franklin system with zero means on $\mathbb{R}$ corresponding to an admissible sequence $\mathcal{T}$. Then $(F_n)_{n=1}^{\infty}$ forms a basis for $H^1(\mathbb{R})$ if and only if $\mathcal{T}$ is 2-regular on $\mathbb{R}$.

Note that the above theorem is a particular case of Theorem 2.1.7.

In Section 2.2 we characterize the sequences of knots for which the Franklin system (F_n) with zero means on $\mathbb{R}$ forms an unconditional basis in the space $H^1(\mathbb{R})$. The main result of Section 2.2 is the following theorem.

Theorem 2.2.2. Let $(F_n)_{n=1}^{\infty}$ be the Franklin system with zero means on $\mathbb{R}$ corresponding to an admissible sequence $\mathcal{T}$. Then $(F_n)_{n=1}^{\infty}$ forms an unconditional basis for $H^1(\mathbb{R})$ if and only if $\mathcal{T}$ is 1-regular on $\mathbb{R}$.

In Chapter 3 we consider questions of convergence of Franklin series.

In Section 3.1 we study the equivalence of absolutely and unconditionally almost everywhere convergence of Franklin series. Let us recall necessary definitions. A functional series $\sum_{n=0}^{\infty} \phi_n(x)$ is said to be unconditionally convergent a.e. on a set E if for every permutation $\{\sigma(n)\}_{n=0}^{\infty}$ of the positive integers the series $\sum_{n=0}^{\infty} \phi_{\sigma(n)}(x)$ converges a.e. on E. A series $\sum_{n=0}^{\infty} \phi_n(x)$ is said to be absolutely convergent a.e. on E if $\sum_{n=0}^{\infty} |\phi_n(x)| < +\infty$ a.e. on E. It is known that a numerical series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent if and only if it is unconditionally convergent. However, for functional series the notions of unconditional convergence a.e. and absolute convergence a.e. are not equivalent since the exceptional set may change under a permutation. For example, if $\{R_n(x)\}_{n=1}^{\infty}$ is a Rademacher system, $\sum_{n=1}^{\infty} a_n^2 < \infty$ and $\sum_{n=1}^{\infty} |a_n| = \infty$, then the series $\sum_{n=1}^{\infty} a_n R_n(x)$ is unconditionally convergent a.e. (see [62]) and absolutely divergent a.e. However, E. Nikishin and P. Ul'yanov [69] proved the following theorem for Haar series.

Theorem 3.1.A. Let $\{\chi_n(x)\}_{n=0}^{\infty}$ be the Haar system. Then the series $\sum_{n=0}^{\infty} a_n\chi_n(x)$ is unconditionally convergent a.e. on a set $E \subset [0; 1]$ if and only if the sum $\sum_{n=0}^{\infty} |a_n\chi_n(x)|$ is finite for a.e. $x \in E$.

G. Gevorkyan extended Theorem 3.1.A to series in the classical Franklin system (see [39]). In [46], Theorem 3.1.A is extended to general Franklin systems that are generated by sequences satisfying some conditions. In Section 3.1 we prove the analogous theorem for all general Franklin systems, without imposing any restrictions.

Theorem 3.1.1. Let $\{f_n(x)\}_{n=0}^{\infty}$ be the Franklin system generated by an admissible sequence $\mathcal{T}$. Then a series $\sum_{n=0}^{\infty} a_n f_n(x)$ is unconditionally convergent a.e. on a set E if and only if it is absolutely convergent a.e. on E.

In Section 3.2 we introduce and study a generalization of Franklin system on $\mathbb{R}$. To define this generalization, we use the same notation, as in the definition of the general Franklin system on

$[0;1]$.

Let $\mathcal{T}=\{t_n : n\geq 0\}$ be an admissible sequence, i.e. $\mathcal{T}$ is dense in $\mathbb{R}$ and $t_i\neq t_j$ if $i\neq j$. For $n\geq 2$, let $\mathcal{T}_n$ be the ordered sequence of points consisting of the grid points $(t_j)_{j=0}^n$, i.e.

$$\mathcal{T}_n=\{\tau_{n,i} : \tau_{n,i}<\tau_{n,i+1}, 0\leq i\leq n\}.$$

We denote by S_n the space of functions defined on $\mathbb{R}$, that are continuous and linear on $[\tau_{n,i};\tau_{n,i+1}]$ and vanish outside $(\tau_{n,0};\tau_{n,n+1})$. It is clear that $\dim S_n=n$ and $S_{n-1}\subset S_n$. Hence there exists a unique (up to sign) function $f\in S_n$, which is orthogonal to S_{n-1} and $\|f\|_2=1$. This function is called the n-th Franklin function on $\mathbb{R}$ corresponding to the partition $\mathcal{T}$.

For a fixed n we denote by $N_{n,i}$, $0\leq i\leq n+1$, the sequence of L^∞-normalized B-splines corresponding to $(\tau_{n,i})_{i=-1}^{n+2}$, where $\tau_{n,-1}:=\tau_{n,0}$ and $\tau_{n,n+2}:=\tau_{n,n+1}$, and having the following properties:

$$\operatorname{supp} N_{n,i}=[\tau_{n,i-1};\tau_{n,i+1}],\quad N_i\geq 0,\quad \sum_{i=0}^{n+1}N_{n,i}(t)=\mathbb{1}_{[\tau_{n,0};\tau_{n,n+1}]}(t).$$

It is clear that the system $\{N_{n,i},\ 1\leq i\leq n\}$ forms a basis of S_n, and $N_{n,i}\notin S_n$ for $i=0$ and $i=n+1$.

Definition 3.2.1. Let $\mathcal{T}$ be an admissible sequence. The general Franklin system $\{f_n(x) : n\geq 1\}$ on $\mathbb{R}$ corresponding to the partition $\mathcal{T}$ is defined as follows: $f_1(x)=\frac{1}{\|N_{1,1}\|_2}N_{1,1}(x)$ and for $n\geq 2$ the function $f_n(x)$ is the n-th Franklin function on $\mathbb{R}$ corresponding to the partition $\mathcal{T}$.

In the study of general Franklin system on $[0;1]$ and orthonormal spline systems the notion of regularity of the sequence $\mathcal{T}$ plays an important role and so in the case of general Franklin system on $\mathbb{R}$.

Since the sequence $\mathcal{T}$ is dense in $\mathbb{R}$, it follows that the system $\{f_n\}_{n=1}^\infty$ is a complete orthonormal system in $L^2(\mathbb{R})$. Essentially repeating the proof of Theorem 2.1 from [48] it can be shown that the Franklin system on $\mathbb{R}$ corresponding to any admissible sequence forms an unconditional basis in the space $L^p(\mathbb{R})$ for any $1<p<\infty$.

In Section 3.2 we find general growth conditions on functions which guarantee the local uniformly convergence of the partial sums of Fourier series with respect to Franklin systems on R for such continuous functions.

Let us give necessary definitions. Let $\varphi(t)$ be an even, positive and increasing function on $[0;\infty)$. Denote

$$C_\varphi(\mathbb{R})=\{f\in C(\mathbb{R}) :\ |f(t)|\leq c\varphi(t)\ \text{ for some } c>0 \text{ and any } t\in\mathbb{R}\}.$$

We construct an admissible sequence $\mathcal{T}^1$ as follows. First we set $t_0=0$, $t_1=-1$, $t_2=1$. Then we add the points $t_3=-2$, $t_4=-\frac{1}{2}$, $t_5=\frac{1}{2}$, $t_6=2$. At the n-th step we take $t_{2^n-1}=-n$, and then successively from left to the right we add the midpoints of the intervals obtained by the points specified up to the n-th step, and set $t_{2^{n+1}-2}=n$. Proceeding inductively, we obtain the desired admissible sequence $\mathcal{T}^1$.

Let the sequence $\mathcal{T}^2$ be constructed by the same algorithm as the sequence $\mathcal{T}^1$, with the only difference that at the n-th step we add the points $t_{2^n-1}=-\xi_n$ and $t_{2^{n+1}-2}=\xi_n$, where $\xi_n\uparrow\infty$.

It is clear that $\mathcal{T}^2$ coincides with $\mathcal{T}^1$ for $\xi_n=n$. Also it is not difficult to see that the sequence $\mathcal{T}^2$ will be 1-regular if and only if

$$0<\inf_n\frac{\xi_{n+2}-\xi_{n+1}}{\xi_{n+1}-\xi_n}\quad\text{and}\quad\sup_n\frac{\xi_{n+2}-\xi_{n+1}}{\xi_{n+1}-\xi_n}<\infty.$$

In particular, the sequence $\mathcal{T}^1$ is 1-regular.

The main results of Section 3.2 are the following theorems.

Theorem 3.2.12. Let $\{f_n(x)\}_{n=1}^{\infty}$ be the Franklin system on $\mathbb{R}$ corresponding to the sequence $\mathcal{T}^2$ and let the function φ satisfy the condition

$$\lim_{n\to\infty} \frac{\ln \varphi(\xi_n)}{2^n} = 0. \tag{6}$$

Then for any function $f \in C_{\varphi}(\mathbb{R})$ the partial sums $S_n(f,t)$ of Fourier series with respect to system $\{f_n(t)\}_{n=1}^{\infty}$ locally uniformly converge to $f(t)$.

Theorem 3.2.13. Let the sequence $\mathcal{T}^2$ be 1-regular with parameter γ and let $\{f_n(x)\}_{n=1}^{\infty}$ be the Franklin system on $\mathbb{R}$ corresponding to the sequence $\mathcal{T}^2$. If the function φ satisfies the condition

$$\limsup_{n\to\infty} \frac{\ln \varphi(\xi_n)}{2^n} > 0, \tag{7}$$

then there exists a function $f \in C_{\varphi}(\mathbb{R})$ such that $S_n(f,t)$ does not converge to $f(t)$ at some points.

Chapter 4 studies the uniqueness problems for Haar, Walsh, Franklin and Ciesielski systems.

In Section 4.1 a reconstruction theorem for additive functions defined on parallelepipeds is obtained. As consequences we derive reconstruction theorems for Haar and Walsh series. Recall a few results obtained in this direction. It is well known from the theory of trigonometric series that the convergence of trigonometric series almost everywhere to zero does not imply that all the coefficients of this series are zero (see, e.g., [93, Chapter IX, Theorem 6.14], [3], [67]). This also applies to other classical systems, for example to series with respect to the Haar, Walsh and Franklin systems.

Uniqueness theorems for almost everywhere convergent or summable trigonometric series were first obtained in the papers [1] and [41], under some additional conditions imposed on the series. These conditions are not stated here, because the results of Chapter 4 concern series with respect to the systems of Haar, Walsh and Franklin as well as their generalizations. Note only that similar problems for multiple trigonometric series were studied in [42], [33].

Let us recall the definition of the A-integral.

Definition 4.1.1. A function $f : [0,1] \to \mathbb{R}$ is said to be A-integrable, if

$$\lim_{\lambda\to\infty} \lambda\, |\{x \in [0,1]; |f(x)| > \lambda\}| = 0,$$

and the following limit exists:

$$\lim_{\lambda\to\infty} \int_0^1 [f(x)]_\lambda dx =: (A) \int_0^1 f(x)dx,$$

where

$$[f(x)]_\lambda = \begin{cases} f(x), & |f(x)| \le \lambda \\ 0, & |f(x)| > \lambda. \end{cases}$$

Let $\{\chi_n(x)\}_{n=1}^{\infty}$ be the Haar system. For series with respect to the Haar system G. Gevorkyan [43] proved, in particular, the following theorem.

Theorem 4.1.A. Let the series with respect to the Haar system

$$\sum_{n=1}^{\infty} a_n \chi_n(x) \tag{8}$$

converge almost everywhere to a function $f(x)$, and let

$$\lim_{\lambda\to\infty} \lambda \left|\{x \in [0,1]; S^*(x) > \lambda\}\right| = 0, \tag{9}$$

where $S^*(x) = \sup_k \left|\sum_{n=1}^{k} a_n \chi_n(x)\right|$ is the majorant of the partial sums of series (8). Then series (8) is a Fourier-Haar series in the sense of A-integration, i.e.

$$a_n = (A)\int_0^1 f(x)\chi_n(x)dx.$$

It is clear that Theorem 4.1.A implies the following:

i) if the series (8) converges a.e. to zero and (9) holds, then all the coefficients a_n are zero;

ii) if the series (8) converges a.e. to a Lebesgue integrable function f and (9) holds, then (8) is the Fourier-Haar series of the function f.

Theorem 4.1.A is a consequence of Theorem 4.1.C, which concerns the uniqueness of additive functions of dyadic cubes.

Then, Theorem 4.1.A was extended by V. Kostin [64] to series with respect to the generalized Haar system and the Price system. Let us introduce necessary definitions.

Suppose that $P = \{p_i\}_{i=1}^{\infty}$ is a sequence of natural numbers distinct from 1, $m_0 = 1$, and $m_j = \prod_{i=1}^{j} p_j$. Then the P-adic expansion of a point $x \in [0,1)$ will have the form

$$x = \sum_{j=1}^{\infty} \frac{x_j}{m_j}, \qquad 0 \le x_j \le p_j - 1,\ x_j \in \mathbb{Z}. \tag{10}$$

To ensure that each point $x \in [0,1)$ is associated with a unique P-adic series for P-adically rational points, i.e. for points of the form l/m_n, where $l = 0, 1, \dots, m_n - 1$, we choose the sums (10) with terminating expansion. Recall the definitions of *multiplicative Price systems* $\{\psi_k(x)\}_{k=0}^{\infty}$ and of *generalized Haar systems.* The Price system is defined by the formulas

$$\psi_k(x) = \exp\left(2\pi i \sum_{j=1}^{n} \frac{\alpha_j x_j}{p_j}\right),$$

where $k = \sum_{j=1}^{n} \alpha_j m_{j-1},\ 0 \le \alpha_j \le p_j - 1,\ j = 1, 2, \dots, n.$

The system of generalized Haar functions $\{\chi_n^P(x)\}_{n=0}^{\infty} =: \{\chi_n(x)\}_{n=0}^{\infty}$ is defined by the formulas

$$\chi_0(x) \equiv 1,$$

and, for $n \ge 1$

$$\chi_n(x) = \chi_{r,s}^{k}(x) = \begin{cases} \sqrt{m_k} e^{2\pi i x_{k+1} s/p_{k+1}}, & x \in \left[\frac{r}{m_k}; \frac{r+1}{m_k}\right) \\ 0, & x \in [0,1)\setminus\left[\frac{r}{m_k}; \frac{r+1}{m_k}\right) \end{cases},$$

where

$$n = m_k + r(p_{k+1} - 1) + s - 1,\ 0 \le r \le m_k - 1,\ 1 \le s \le p_{k+1} - 1.$$

Note that the Haar system corresponds to the sequence $p_i = 2,\ i \in \mathbb{N}$.

For such systems V. Kostin [64] proved the following theorem.

Theorem 4.1.B. Let

$$\sum_{n=0}^{\infty} a_n \varphi_n(x)$$

be a series with respect to either the Price system or the generalized Haar system with

$$\sup_{n\in\mathbb{N}} p_n < \infty.$$

If the sequence of partial sums $S_{m_n}(x) := \sum_{j=0}^{m_n-1} a_j \varphi_j(x)$ of the series converges almost everywhere to a function $f(x)$ and there exists a sequence $\{\lambda_m\}_{m=1}^{\infty}$ for which

$$\lim_{m\to\infty} \lambda_m |\{x \in [0,1]; S^*(x) > \lambda_m\}| = 0,$$

where $S^*(x) = \sup_{n\in\mathbb{N}} |S_{m_n}(x)|$, then the nth coefficient of the series under consideration can be recovered by the formula

$$a_n = \lim_{m\to\infty} \int_0^1 [f(x)]_{\lambda_m} \overline{\varphi_n(x)} dx.$$

Analogues of Theorems 4.1.A, 4.1.B for Stromberg system has been obtained in [6*].

Here we strengthen these theorems by considering the wider $A\mathcal{H}$-integral introduced by K. Yoneda [91].

In order to state the results we first introduce some necessary notation and definitions. For a complex-valued function $f(x)$ and non-negative function $\lambda(x)$ we define the truncation of f as follows:

$$[f(x)]_{\lambda(x)} := \begin{cases} f(x), & |f(x)| \le \lambda(x) \\ 0, & |f(x)| > \lambda(x) \end{cases}.$$

Definition 4.1.2.([91],[92]) Let $\mathcal{H} = \{h_n(x)\}_{n=1}^{\infty}$ be a sequence of real-valued Lebesgue integrable functions on $[0,1]$ such that a.e.

$$0 \le h_1(x) \le h_2(x) \le \cdots \le h_n(x) \le \dots,\ \lim_{n\to\infty} h_n(x) = \infty. \tag{11}$$

Then $f : [0,1] \to C$ is said to be $A\mathcal{H}$-integrable on $[0,1]$, if

$$\lim_{n\to\infty} \int_{\{x\in[0,1];|f(x)|\ge\alpha h_n(x)\}} h_n(x) dx = 0,\ \text{for all } \alpha > 0$$

and the following limit exists

$$\lim_{n\to\infty} \int_0^1 [f(x)]_{h_n(x)} dx =: (A\mathcal{H}) \int_0^1 f(x) dx,$$

called the $A\mathcal{H}$-integral of the function $f(x)$.

If $h_n(x) \equiv n$, then we obtain the ordinary A-integral, but if $h_n(x) \equiv \lambda_n$, where $\Lambda := \{\lambda_n\}_{n=1}^{\infty}$ is an increasing sequence, which tends to infinity, then we obtain a generalization of the A-integral.

Similarly, we can extend the definition of the $A\mathcal{H}$-integral to complex-valued functions defined on a parallelepiped $I \subset \mathbb{R}^n$.

Remark 4.1.3. If, for the function $f : I \to C$, there exists a sequence of functions $\mathcal{H} = \{h_n(\mathbf{x})\}_{n=1}^{\infty}$ satisfying (11) such that

$$\lim_{n\to\infty} \int_{\{\mathbf{x}\in I;|f(\mathbf{x})|\geq h_n(\mathbf{x})\}} h_n(\mathbf{x})d\mathbf{x} = 0, \tag{12}$$

and the limit

$$\lim_{n\to\infty} \int_I [f(\mathbf{x})]_{h_n(\mathbf{x})} d\mathbf{x}$$

exists, then there exists a sequence of functions $\mathcal{G}$ such that the function $f(\mathbf{x})$ is $A\mathcal{G}$-integrable and

$$\lim_{n\to\infty} \int_I [f(\mathbf{x})]_{h_n(\mathbf{x})} d\mathbf{x} = (A\mathcal{G}) \int_I f(\mathbf{x})d\mathbf{x}. \tag{13}$$

Indeed, it follows from condition (12) that there exists a non-decreasing sequence $\{\alpha_n\}_{n=1}^{\infty}$, tending to infinity such that

$$\lim_{n\to\infty} \alpha_n \int_{\{\mathbf{x}\in I;|f(\mathbf{x})|\geq h_n(\mathbf{x})\}} h_n(\mathbf{x})d\mathbf{x} = 0. \tag{14}$$

Denoting $g_n(\mathbf{x}) = \alpha_n h_n(\mathbf{x})$, we find that for any $\alpha > 0$ the inequality $\alpha g_n(\mathbf{x}) > h_n(\mathbf{x})$ holds for sufficiently large n and therefore

$$0 \leq \lim_{n\to\infty} \int_{\{\mathbf{x}\in I;|f(\mathbf{x})|\geq \alpha g_n(\mathbf{x})\}} g_n(\mathbf{x})d\mathbf{x} \leq \lim_{n\to\infty} \alpha_n \int_{\{\mathbf{x}\in I;|f(\mathbf{x})|\geq h_n(\mathbf{x})\}} h_n(\mathbf{x})d\mathbf{x} = 0.$$

In addition, from (14) we obtain

$$\left| \int_I [f(\mathbf{x})]_{h_n(\mathbf{x})} d\mathbf{x} - \int_I [f(\mathbf{x})]_{g_n(\mathbf{x})} d\mathbf{x} \right| \leq \int_{\{\mathbf{x}\in I;|f(\mathbf{x})|\geq h_n(\mathbf{x})\}} g_n(\mathbf{x})d\mathbf{x} \to 0,$$

and therefore equality (13) holds.

Let $P^j = \{p_i^j\}_{i=1}^{\infty}$, $1 \leq j \leq d$ be sequences of natural numbers distinct from 1 and $\mathcal{P} = \{P^j\}_{j=1}^{d}$. Set

$$m_0^j = 1 \ \text{ and } \ m_k^j = \prod_{i=1}^{k} p_i^j.$$

Denote by Λ_k^d the set of all $\mathcal{P}$-adic parallelepipeds of rank k:

$$\Lambda_k^d = \left\{ I; I = \left[\frac{n_1}{m_k^1}, \frac{n_1+1}{m_k^1}\right] \times \ldots \times \left[\frac{n_d}{m_k^d}, \frac{n_d+1}{m_k^d}\right], n_1, \ldots, n_d \in \mathbb{Z} \right\},$$

as well as write $r(I) = k$, if $I \in \Lambda_k^d$.

Let

$$\Lambda^d = \cup_{k\in\mathbb{N}_0} \Lambda_k^d.$$

Here $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{N}$ stands for the set of natural numbers.

A complex-valued function Ψ defined on the set Λ^d will be called an *additive function of $\mathcal{P}$-adic parallelepipeds* if, for all $I, I_1, \dots, I_n \in \Lambda^d$, for which $I = \cup_{i=1}^n I_i$ and $\operatorname{int}(I_i) \cap \operatorname{int}(I_j) = \emptyset, i \neq j$, the following equality holds:

$$\Psi(I) = \sum_{i=1}^{n} \Psi(I_i).$$

A point $x \in \mathbb{R}$ is said to be *irrational with respect to the sequence* $\{p_i\}_{i=1}^{\infty}$, if $x \cdot \prod_{i=1}^{n} p_i \notin \mathbb{Z}$, for all $n \in \mathbb{N}$. A point $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ is said to be *$\mathcal{P}$-adically irrational* if, for all $1 \leq j \leq d$, the point x_j is irrational with respect to the sequence P^j.

For $\mathcal{P}$-adically irrational points $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, we define the derivative $\Psi'(\mathbf{x})$ and the majorant $\Psi^*(\mathbf{x})$ as follows:

$$\Psi'(\mathbf{x}) = \lim_{\substack{k\to\infty \\ \mathbf{x}\in I_k \in \Lambda_k^d}} \frac{\Psi(I_k)}{|I_k|}, \quad \Psi^*(\mathbf{x}) = \sup_{I:\mathbf{x}\in I\in\Lambda^d} \frac{|\Psi(I)|}{|I|}.$$

Fix $I_0 \in \Lambda^d$. Let the functions $h_m : I_0 \to \mathbb{R},\ m \in \mathbb{N}$ satisfy the following conditions:

$$\text{(i)} \quad 0 \leq h_1(\mathbf{x}) \leq h_2(\mathbf{x}) \leq \dots \leq h_m(\mathbf{x}) \leq \dots, \ \lim_{m\to\infty} h_m(\mathbf{x}) = \infty, \tag{15}$$

(ii) there exists a constant $C > 0$ and, for each $m \in \mathbb{N}$, parallelepipeds $I_1^m, \dots, I_{n_m}^m \in \Lambda^d$ such that $\operatorname{int}(I_i^m) \cap \operatorname{int}(I_j^m) = \emptyset, i \neq j, \quad \cup_{k=1}^{n_m} I_k^m = I_0$ and

$$\sup_{\mathbf{x}\in I_k^m} h_m(\mathbf{x}) \leq C \inf_{\mathbf{x}\in I_k^m} h_m(\mathbf{x}), \tag{16}$$

for all $m \in \mathbb{N},\ 1 \leq k \leq n_m$, and

$$\text{(iii)} \quad \inf_{m,k} \int_{I_k^m} h_m(\mathbf{x}) d\mathbf{x} > 0. \tag{17}$$

In other words, for any function h_m the parallelepiped I_0 can be split into small $\mathcal{P}$-adic parallelepipeds, so that the supremum and infimum of that function on each parallelepiped are comparable and integrals over that parallelepipeds are bounded away from 0.

One of the main result of Section 4.1 is the following:

Theorem 4.1.4. Let the sequence of functions $h_m(\mathbf{x})$ satisfy conditions (15)-(17), and let Ψ be a complex-valued additive function defined on $\mathcal{P}$-adic parallelepipeds, where $\mathcal{P} = \{P^j\}_{j=1}^d, P^j = \{p_i^j\}_{i=1}^{\infty}$ and

$$\sup_{i\in\mathbb{N}, 1\leq j\leq d} p_i^j < \infty. \tag{18}$$

If

$$\lim_{m\to\infty} \int_{\{\mathbf{x}\in I_0; \Psi^*(\mathbf{x}) > h_m(\mathbf{x})\}} h_m(\mathbf{x}) d\mathbf{x} = 0 \tag{19}$$

and $\Psi'(\mathbf{x})$ exists a.e., then, for any $I \in \Lambda^d$, $I \subset I_0$, the following equality holds:

$$\Psi(I) = \lim_{m\to\infty} \int_I [\Psi'(\mathbf{x})]_{h_m(\mathbf{x})}\, d\mathbf{x}.$$

Taking $p_i^j = 2$ for all $i \in \mathbb{N}, 1 \leq j \leq d$, and the functions $h_m(\mathbf{x}) \equiv \lambda_m$ in Theorem 4.1.4 we obtain the following theorem proved by G. Gevorkyan in [43].

Theorem 4.1.C. Let Φ be an additive function defined on dyadic cubes. If there exists a sequence λ_m tending to infinity such that

$$\lim_{m\to\infty} \lambda_m |\{\mathbf{x} \in I_0; \Phi^*(\mathbf{x}) > \lambda_m\}| = 0, \tag{20}$$

and the derivative $\Phi'(\mathbf{x})$ exists a.e., then, for any dyadic cube $I \subset I_0$ the following equality holds:

$$\Phi(I) = \lim_{m\to\infty} \int_I [\Phi'(\mathbf{x})]_{\lambda_m}\, d\mathbf{x}.$$

In Section 4.1 we give an example of additive function of dyadic cubes which can be recovered from its derivative applying Theorem 4.1.4, but not Theorem 4.1.C.

Now let us give applications of Theorem 4.1.4 to Haar and Walsh series. For any $\mathcal{P} = \{P^j\}_{j=1}^d$ we consider a multiple series

$$\sum_{\mathbf{n}\in\mathbb{N}_0^d} a_{\mathbf{n}} \chi_{\mathbf{n}}^{\mathcal{P}}(\mathbf{x}) = \sum_{\substack{n_j\in\mathbb{N}_0\\ j=1,\dots,d}} a_{n_1\dots n_d} \chi_{n_1}^{P^1}(x_1)\dots\chi_{n_d}^{P^d}(x_d) \tag{21}$$

and the $\mathcal{P}$-adic partial sums of this series, i.e.

$$\sum_{\mathbf{n}<\widetilde{\mathbf{N}}} a_{\mathbf{n}} \chi_{\mathbf{n}}^{\mathcal{P}}(\mathbf{x}),$$

where by $\mathbf{n} < \mathbf{k}$ we mean $0 \le n_j < m_{k_j}^j$, for all $j = 1, 2, \dots, d$, and $\widetilde{\mathbf{N}} = (N, \dots, N) \in \mathbb{R}^d$, for any natural number N. It is easy to see that there exists a series with respect to the Price system

$$\sum_{\mathbf{n}\in\mathbb{N}_0^d} b_{\mathbf{n}} \psi_{\mathbf{n}}^{\mathcal{P}}(\mathbf{x}) \tag{22}$$

such that, for all $k \in \mathbb{N}^d$, the following equality holds:

$$\sum_{\mathbf{n}<\mathbf{k}} a_{\mathbf{n}} \chi_{\mathbf{n}}^{\mathcal{P}}(\mathbf{x}) = \sum_{\mathbf{n}<\mathbf{k}} b_{\mathbf{n}} \psi_{\mathbf{n}}^{\mathcal{P}}(\mathbf{x}). \tag{23}$$

In particular, this implies that the $\mathcal{P}$-adic partial sums of this series coincide with the $\mathcal{P}$-adic partial sums of the series (21):

$$S_N^{\chi}(\mathbf{x}) := \sum_{\mathbf{n}<\widetilde{\mathbf{N}}} a_{\mathbf{n}} \chi_{\mathbf{n}}^{\mathcal{P}}(\mathbf{x}) = \sum_{\mathbf{n}<\widetilde{\mathbf{N}}} b_{\mathbf{n}} \psi_{\mathbf{n}}^{\mathcal{P}}(\mathbf{x}) =: S_N^{\psi}(\mathbf{x}). \tag{24}$$

Thus each Haar series (21) can be associated with a Price series (22), and vice versa. To each series (21) let us assign the additive function Ψ as follows:

$$\Psi(I) = \lim_{N\to\infty} \int_I \sum_{\mathbf{n}<\widetilde{\mathbf{N}}} a_{\mathbf{n}} \chi_{\mathbf{n}}^{\mathcal{P}}(\mathbf{x}) d\mathbf{x}, \text{ for } I \in \Lambda^d. \tag{25}$$

The right-hand limit exists for all $I \in \Lambda^d$, because for a sufficiently large N (depending on I) we have

$$\int_I a_{\mathbf{n}} \chi_{\mathbf{n}}^{\mathcal{P}}(\mathbf{x}) d\mathbf{x} = 0, \text{ for } \max_{1\le j\le d} \left\{ \frac{n_j}{m_N^j} \right\} \ge 1.$$

It follows from (25) that, for each $\mathcal{P}$-adically irrational point $\mathbf{x}$,

$$\Psi'(\mathbf{x}) = \lim_{N\to\infty} \sum_{\mathbf{n}<\widetilde{\mathbf{N}}} a_{\mathbf{n}} \chi_{\mathbf{n}}^{\mathcal{P}}(\mathbf{x}) \quad \text{and} \quad \Psi^*(\mathbf{x}) = \sup_N \left| \sum_{\mathbf{n}<\widetilde{\mathbf{N}}} a_{\mathbf{n}} \chi_{\mathbf{n}}^{\mathcal{P}}(\mathbf{x}) \right|. \tag{26}$$

Applying Theorem 4.1.4 we obtain the next result.

Theorem 4.1.7. Let the $\mathcal{P}$-adic sums of the multiple Haar series (21) converge a.e. to a function $f(\mathbf{x})$, and let $\mathcal{P}$ satisfy (18). If there exists a sequence of functions $\{h_m(\mathbf{x})\}_{m=1}^{\infty}$ satisfying conditions (15)-(17), for which

$$\lim_{m\to\infty} \int_{\{\mathbf{x}\in[0,1]^d;\Psi^*(\mathbf{x})>h_m(\mathbf{x})\}} h_m(\mathbf{x}) d\mathbf{x} = 0,$$

then for any $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$,

$$a_{\mathbf{n}} = \lim_{m\to\infty} \int_{[0,1]^d} [f(\mathbf{x})]_{h_m(\mathbf{x})} \overline{\chi_{\mathbf{n}}^{\mathcal{P}}(\mathbf{x})} d\mathbf{x}.$$

Note that the equality of the corresponding $\mathcal{P}$-adic partial sums of the series (24) and the definition of $\Psi(I)$ imply that

$$\Psi(I) = \lim_{N\to\infty} \int_I S_N^{\psi}(\mathbf{x}) d\mathbf{x},$$

also, from (24) and (26) for each $\mathcal{P}$-adically irrational point $\mathbf{x}$ we will have

$$\Psi'(\mathbf{x}) = \lim_{N\to\infty} S_N^{\psi}(\mathbf{x}) \quad \text{and} \quad \Psi^*(\mathbf{x}) = \sup_N |S_N^{\psi}(\mathbf{x})|.$$

A similar theorem can be proved also for the Price system. Namely, the following theorem is valid.

Theorem 4.1.8. Let the $\mathcal{P}$-adic partial sums of the series (22) converge a.e. to a function $f(\mathbf{x})$, and let $\mathcal{P}$ satisfies (18). If there exists a sequence of functions $\{h_m(\mathbf{x})\}_{m=1}^{\infty}$ satisfying conditions (15)-(17) for which

$$\lim_{m\to\infty} \int_{\{\mathbf{x}\in[0,1]^d;\Psi^*(\mathbf{x})>h_m(\mathbf{x})\}} h_m(\mathbf{x}) d\mathbf{x} = 0,$$

then, for any $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$

$$b_{\mathbf{n}} = \lim_{m\to\infty} \int_{[0,1]^d} [f(\mathbf{x})]_{h_m(\mathbf{x})} \overline{\psi_{\mathbf{n}}^{\mathcal{P}}(\mathbf{x})} d\mathbf{x}.$$

In Theorems 4.1.7 and 4.1.8, taking $d = 1$ and $h_m(x) \equiv \lambda_m$, we obtain Theorem 4.1.B proved by V. Kostin in [64].

In Section 4.2 we study the uniqueness of series in the classical Franklin system $\{\mathbf{f}_n(x)\}_n$. The main purpose of Section 4.2 is to prove a restoration theorem for series in the Franklin system, similar to Theorem 4.1.7, which is for series in the Haar system.

To the best of our knowledge, the first statement on the uniqueness of series in the Franklin system was the uniqueness theorem stated in [36]. The following theorem is proved in [36].

Theorem 4.2.A. Let the series

$$\sum_{n=0}^{\infty} a_n \mathbf{f}_n(x), \tag{27}$$

with coefficients $a_n = o(\sqrt{n})$, converge in measure to a bounded function f and converge everywhere, except, possibly, some countable set,

$$\sup_N \max_{t\in\left(x-\frac{1}{N};x+\frac{1}{N}\right)} \left|\sum_{n=0}^{N} a_n \mathbf{f}_n(t)\right| < +\infty.$$

Then the series (27) is the Fourier-Franklin series of the function f.

For series in the Franklin system, uniqueness theorems of another type were proved; in such theorems, one necessary condition on the majorant of the partial sums of the series is imposed (see [38, Theorem 3]).

Theorem 4.2.B. The series (27) is the Fourier-Franklin series of an integrable function f if and only if this series a.e. converge to $f(x)$ and

$$\liminf_{\lambda\to\infty} \lambda \cdot \left|\left\{x \in [0;1] : \sup_N \left|\sum_{n=0}^{N} a_n \mathbf{f}_n(x)\right| > \lambda\right\}\right| = 0.$$

A similar theorem for the general Franklin system was proved by M. Poghosyan in [76].

G. Gevorgyan and M. Poghosyan [53] obtained a coefficients recovering theorem for multiple Franklin series under the condition

$$\liminf_{\lambda\to\infty} \lambda |\{\mathbf{x} \in [0,1]^k; S^*(\mathbf{x}) > \lambda\}| = 0,$$

where $S^*(\mathbf{x})$ is the majorant of 2^nth cubic partial sums.

To formulate the main results of Section 4.2, we shall give some definitions.

Let $\{h_m(x)\}$ be a sequence of functions $h_m(x) : [0,1] \to R$, satisfying the following conditions:

$$\text{(i) } 0 \le h_1(x) \le h_2(x) \le \dots \le h_m(x) \le \dots, \quad \lim_{m\to\infty} h_m(x) = \infty, \tag{28}$$

and there exist dyadic points $0 = t_{m,0} < t_{m,1} < \ldots < t_{m,n_m} = 1$ such that the intervals $I_k^m = [t_{m,k-1}, t_{m,k})$, $k = 1, \ldots, n_m$, are dyadic, that is,

$$I_k^m \in \mathcal{D} := \left\{ \left[\frac{i}{2^j}, \frac{i+1}{2^j}\right]; 0 \le i \le 2^j - 1, j \ge 0 \right\},$$

and the function $h_m(x)$ is constant on these intervals:

$$\text{(ii) } h_m(x) = \lambda_k^m \text{ for } x \in I_k^m, k = 1, \ldots, n_m, \tag{29}$$

besides, the following conditions are fulfilled:

$$\text{(iii) } \inf_{m,k} \int_{I_k^m} h_m(x)dx = \inf_{m,k} |I_k^m| \lambda_k^m > 0, \tag{30}$$

$$\text{(iv) } \sup_{m,k} \left(\frac{\lambda_k^m}{\lambda_{k-1}^m} + \frac{\lambda_{k-1}^m}{\lambda_k^m}\right) < +\infty. \tag{31}$$

In other words, for each function $h_m(x)$, the interval $[0,1]$ can be split into dyadic intervals so that on each of these dyadic intervals the function takes values that are equivalent to the values of the function on the adjacent intervals, and the integrals over these intervals are greater than some positive constant.

It turns out that if the functions h_m satisfy conditions (28)-(31), then one can choose new dyadic intervals I_k^m so that together with conditions (30), (31), the following condition also will be satisfied:

$$\sup_{m,k}\left(\frac{|I_k^m|}{|I_{k-1}^m|}+\frac{|I_{k-1}^m|}{|I_k^m|}\right)<+\infty. \tag{32}$$

Consider the series $\sum_{n=0}^{\infty} a_n \mathbf{f}_n(x)$, and denote

$$\sigma_\nu(x) := \sum_{n=0}^{2^\nu} a_n \mathbf{f}_n(x), \quad \sigma^*(x) := \sup_\nu |\sigma_\nu(x)|.$$

Now we are able to state the main results of Section 4.2.

Theorem 4.2.1. Let the sequence of functions $h_m(x)$ satisfy the conditions (28)-(31), the sequence of partial sums σ_ν converges in measure to a function f, and let the majorant σ^* of partial sums σ_ν satisfy the following condition:

$$\lim_{m\to\infty}\int_{\{x\in[0,1];\ \sigma^*(x)>h_m(x)\}} h_m(x)dx = 0. \tag{33}$$

Then for all $n \geq 0$ we have

$$a_n = \lim_{m\to\infty}\int_0^1 [f(x)]_{h_m(x)} \mathbf{f}_n(x)dx. \tag{34}$$

The next theorem to some extent shows the necessity of condition (31) in Theorem 4.2.1.

Theorem 4.2.2. Let functions $h_m(x)$ satisfy conditions (28)-(30), (32) and

$$\sup_{m,k}\left(\frac{\lambda_k^m}{\lambda_{k-1}^m\cdot\log\lambda_k^m}+\frac{\lambda_{k-1}^m}{\lambda_k^m\cdot\log\lambda_{k-1}^m}\right)=+\infty. \tag{35}$$

Then there exists a series $\sum_{n=0}^{\infty} a_n \mathbf{f}_n$, converging a.e. to a function f with a majorant σ^* satisfying (33), but the coefficients $a_n, n \geq 0$ are not recovered by formulas (34). In particular,

$$\limsup_{m\to\infty}\int_0^1 [f(x)]_{h_m(x)} \mathbf{f}_0(x)dx = \limsup_{m\to\infty}\int_0^1 [f(x)]_{h_m(x)}\, dx = +\infty.$$

In Section 4.3 we prove uniqueness results for series in the Ciesielski system, which was first introduced in [13]. Recall that the Ciesielski system $\{\mathbf{f}^{(r)}\}_{n=-r+2}^{\infty}$ is the orthonormal spline system of order r corresponding to the dyadic sequence. Note that the case $r = 2$ corresponds to the orthonormal system of piecewise linear functions, i.e. to the classical Franklin system.

Multiple series in the Franklin system were studied in [44]. Let k be a natural number. Consider a multiple Franklin series

$$\sum_{\mathbf{m}\in\mathbb{N}_0^k} a_{\mathbf{m}}\mathbf{f}_{\mathbf{m}}(\mathbf{x}), \tag{36}$$

where $\mathbf{m} = (m_1, ..., m_k) \in \mathbb{N}_0^k$ is a vector with integer coordinates, $\mathbf{x} = (x_1, ..., x_k) \in [0;1]^k$ and $\mathbf{f_m}(\mathbf{x}) = \mathbf{f}_{m_1}(x_1) \cdots \mathbf{f}_{m_k}(x_k)$. It is said that a series (36) is rectangularly convergent at the point $\mathbf{x}$ if the limit

$$\lim_{\mathbf{M}\to+\infty} \sum_{\mathbf{m}\le\mathbf{M}} a_{\mathbf{m}} \mathbf{f_m}(\mathbf{x}),$$

exists; here $\mathbf{m} \le \mathbf{M}$ means that $m_j \le M_j$, $j = 1, ..., k$, and $\mathbf{M} = (M_1, ..., M_k) \to +\infty$ means that $\min_j M_j \to +\infty$.

The following theorem holds (see [44, Theorem 10]).

Theorem 4.3.A. The series (36) is the Fourier-Franklin series of a function $f \in L \ln^{k-1} L([0;1]^k)$ if and only if the series (36) a.e. rectangularly converges to $f(\mathbf{x})$ and

$$\liminf_{\lambda\to+\infty} \lambda \cdot \left| \left\{ \mathbf{x} \in [0;1]^k : \sup_{\mathbf{M}} \left| \sum_{\mathbf{m}\le\mathbf{M}} a_{\mathbf{m}} \mathbf{f_m}(\mathbf{x}) \right| > \lambda \right\} \right| = 0.$$

Recently G. Gevorkyan [54] essentially strengthened Theorems 4.2.B and 4.3.A by using methods distinct from those applied in [44].

Theorem 4.3.B.([54]) The series (36) is the Fourier-Franklin series of a function $f \in L([0;1]^k)$ if and only if the following conditions hold:

1. the sums $\sum\limits_{\mathbf{m}: m_i \le 2^\nu} a_{\mathbf{m}} \mathbf{f_m}(\mathbf{x})$ converge in measure to f;
2. $\liminf\limits_{\lambda\to+\infty} \lambda \cdot \left| \left\{ \mathbf{x} \in [0;1]^k : \sup_\nu \left| \sum\limits_{\mathbf{m}: m_i \le 2^\nu} a_{\mathbf{m}} \mathbf{f_m}(\mathbf{x}) \right| > \lambda \right\} \right| = 0.$

Consider a multiple Ciesielski series

$$\sum_{\mathbf{m}: m_i \ge -r+2} a_{\mathbf{m}} \mathbf{f}^{(r)}_{\mathbf{m}}(\mathbf{x}). \tag{37}$$

Denote by $\sigma_\nu(\mathbf{x})$ the 2^νth cubic partial sums of the series (37), i.e.

$$\sigma_\nu(\mathbf{x}) = \sum_{\mathbf{m}: m_i \le 2^\nu} a_{\mathbf{m}} \mathbf{f}^{(r)}_{\mathbf{m}}(\mathbf{x}). \tag{38}$$

Set $\sigma^*(\mathbf{x}) = \sup_\nu |\sigma_\nu(\mathbf{x})|$. The following theorems hold.

Theorem 4.3.1. If the sums (38) converge in measure to zero and the following condition holds:

$$\liminf_{\lambda\to+\infty} \lambda \cdot \left| \left\{ \mathbf{x} \in [0;1]^k : \sigma^*(\mathbf{x}) > \lambda \right\} \right| = 0, \tag{39}$$

then all the coefficients of the series (37) are zero.

Theorem 4.3.2. If the series (37) is the Fourier-Ciesielski series of a function $f \in L([0;1]^k)$, then the sums (38) converge a.e. to f and

$$\lim_{\lambda\to+\infty} \lambda \cdot \left| \left\{ \mathbf{x} \in [0;1]^k : \sigma^*(\mathbf{x}) > \lambda \right\} \right| = 0. \tag{40}$$

Theorems 4.3.1 and 4.3.2 imply the following theorem.

Theorem 4.3.3. The series (37) is the Fourier-Ciesielski series of a function $f \in L([0;1]^k)$ if and only if the following conditions hold:

1. the sums $\sigma_\nu(\mathbf{x})$ converge in measure to f;

2. $\liminf_{\lambda\to+\infty} \lambda \cdot \left|\left\{\mathbf{x} \in [0;1]^k : \sup_\nu |\sigma_\nu(\mathbf{x})| > \lambda\right\}\right| = 0.$

We have already stated several uniqueness theorems for Haar series (see Theorems 4.1.1, 4.1.B and 4.1.7). Note that the partial sums of a Haar series are piecewise constant functions. In Section 4.4 we are interested in obtaining a result analogous to Theorem 4.1.7 for univariate piecewise polynomial sequences.

In order to formulate the result let us give some necessary definitions.

Let $r \in \mathbb{N}$. Denote by $\mathcal{S}_n^{(r)}$ the space of piecewise polynomial functions whose restrictions on each $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$ for $0 \le k \le 2^n - 1$, are polynomials of degree not exceeding r, i.e.

$$\mathcal{S}_n^{(r)} = \left\{ f; \deg(f|_{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)}) \le r \text{ for } 0 \le k \le 2^n - 1 \right\}.$$

Let $\mathcal{P}_n^{(r)} : L[0,1] \to \mathcal{S}_n^{(r)}$ be the orthogonal projection, i.e.

$$(f, g) = (\mathcal{P}_n^{(r)} f, g) \text{ for all } f \in L[0,1],\ g \in \mathcal{S}_n^{(r)}.$$

Let a sequence of functions $(S_n)_{n\ge 0}$ satisfy $S_n \in \mathcal{S}_n^{(r)}$ for $n \ge 0$ and

$$\mathcal{P}_n^{(r)}(S_m) = S_n \text{ for } m \ge n. \tag{41}$$

Set $S^*(x) = \sup_n |S_n(x)|$. We denote by $\mathcal{D}$ the set of all dyadic intervals, i.e.

$$\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n, \quad \text{where} \quad \mathcal{D}_n = \left\{ \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) ; 0 \le k \le 2^n - 1 \right\}.$$

We call an interval $I \in \mathcal{D}_n$ an interval of rank n and set $r(I) := n$.

Let functions $h_m(x),\ h_m : [0,1] \to R$ satisfy the following conditions:

$$\text{(i) } 0 \le h_1(x) \le h_2(x) \le \dots \le h_m(x) \le \dots,\ \lim_{m\to\infty} h_m(x) = \infty, \tag{42}$$

(ii) there exist a constant $C > 0$ and intervals $I_1^m, \dots, I_{n_m}^m \in \mathcal{D}$ so that $I_i^m \cap I_j^m = \emptyset, i \ne j$, $\cup_{k=1}^{n_m} I_k^m = [0,1)$, and

$$\sup_{x\in I_k^m} h_m(x) \le C \inf_{x\in I_k^m} h_m(x), \tag{43}$$

for any $m \in \mathbb{N},\ 1 \le k \le n_m$, and

$$\text{(iii) } \inf_{m,k} \int_{I_k^m} h_m(x)dx > 0. \tag{44}$$

In other words, for any function h_m the interval $[0,1]$ can be split into small dyadic intervals, so that the supremum and infimum of that function on each interval are comparable and integrals over that intervals are bounded away from zero.

Theorem 4.4.1. Let the functions $h_m(x)$ satisfy conditions (42), (43), (44). If the sequence (S_n), satisfying (41), converges in measure to a function S and $\lim_{m\to\infty} \int_{\{x\in[0,1]; S^*(x) > h_m(x)\}} h_m(x)dx = 0$, then for any $g \in \mathcal{S}_n^{(r)}$ we have

$$(S_n, g) = \lim_{m\to\infty} \int_0^1 [S(x)]_{h_m(x)}\, g(x)dx.$$

This theorem actually enables to recover the sequence (S_n) from its limit S under the mentioned conditions. Generally speaking, the limit may not be Lebesgue integrable.

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Ամփոփում

Ատենախոսությունում հետազոտվում են տարբեր սպլայն համակարգերի բազիսության, զուգամիտության և միակության հարցեր:

Առաջին գլխում ստացվել են վերջնական նկարագրություններ բոլոր այն տրոհումների, որոնց համապատասխան Ֆրանկլինի պարբերական համակարգը բազիս կամ ոչ պայմանական բազիս է Լեբեգի, Հարդիի կամ Գ. Սոուզայի [23,24] կողմից ներմուծված տարածություններում: Նշենք, որ չնայած նրան, որ վերջին տարածությունը հանդիսանում է Հարդիի տարածության ենթատարածություն, Ֆրանկլինի պարբերական համակարգը կամ միաժամանակ բազիս (ոչ պայմանական բազիս) է այդ երկու տարածություններում, կամ էլ այն չի հանդիսանում բազիս (ոչ պայմանական բազիս) այդ տարածություններից և ոչ մեկի համար: Ստացվել է նաև երկրաչափական նկարագրություն բոլոր այն տրոհումների, որոնց համապատասխան բարձր կարգի սպլայններից կազմված օրթոնորմալ համակարգը ոչ պայմանական բազիս է Հարդիի տարածությունում: Ինչպես Ֆրանկլինի պարբերական համակարգի, այնպես էլ օրթոնորմալ սպլայն համակարգի Հարդիի տարածությունում ոչ պայմանական բազիսության ապացույցը հենված է մասնավորապես համապատասխան Պելիի քառակուսային ֆունկցիայի և մասնակի գումարների մաժորանտի L^1 նորմերի համարժեքության վրա: Մենք ցույց ենք տվել, որ Ֆրանկլինի դասական համակարգով շարքերի Պելիի քառակուսային ֆունկցիայի և մասնակի գումարների մաժորանտի կամայական երկուական հատվածով տարածված L^p նորմերի համարժեքությունը բոլոր p դրական թվերի համար որոշ պայմանի դեպքում: Բացի այդ, ցույց է տրվել նաև այդ պայմանի անհրաժեշտությունը:

Երկրորդ գլխում ուսումնասիրվում է առանցքի վրա զրո միջինով սպլայն համակարգերի բազիսության հարցը Հարդիի տարածությունում: Ստացվել են անհրաժեշտ և բավարար պայմաններ առանցքի վրա զրո միջինով կամայական ֆիքսած կարգի սպլայն համակարգը ծնող տրոհման համար, որոնց դեպքում համապատասխան համակարգը բազիս է Հարդիի տարածությունում: Ստացված պայմանները տարբերվում են Հարդիի տարածությունում միավոր հատվածի վրա որոշված օրթոնորմալ սպլայն համակարգերի բազիսության համար անհրաժեշտ և բավարար պայմաններից: Առանցքի վրա զրո միջինով Ֆրանկլինի համակարգերի համար ստացվել են նաև Հարդիի տարածությունում ոչ պայմանական բազիսության համար անհրաժեշտ և բավարար պայմաններ:

Երրորդ գլուխը նվիրված է զուգամիտության հարցերին: Ապացուցվել է, որ Ֆրանկլինի

ընդհանուր համակարգով շարքը համարյա ամենուրեք ոչ պայմանական զուգամետ է որոշ բազմության վրա, այն և միայն այն դեպքում երբ այդ շարքը համարյա ամենուրեք բացարձակ զուգամետ է նույն բազմության վրա: Նաև կառուցվել են երկրորդ կարգի սպլայն համակարգեր, որ կամայական անընդհատ և նախապես ֆիքսած ֆունկցիան չգերազանցող ֆունկցիայի Ֆուրիեի շարքը ըստ այդ համակարգի լոկալ հավասարաչափ զուգամետ է:

Չորրորդ գլխում ստացվել են միակությանը վերաբերող արդյունքներ: Ապացուցվել է, որ P-ադիկ զուգահեռանիստների վրա ադիտիվ ֆունկցիան կարելի է վերականգնել իր ածանցյալից, եթե նրա մաժորանտը բավարարում է որոշակի պայմանի: Կիրառելով այս արդյունքը ստացվել են Հաարի և Ուոլշի ընդհանրացրած համակարգերով բազմապատիկ շարքերի գործակիցների վերականգման բանաձևեր այդ շարքերի գումարի միջոցով, եթե շարքի մաժորանտը բավարարում է որոշակի պայմանի: Նմանատիպ թեորեմ ստացվել է նաև Ֆրանկլինի դասական համակարգով շարքերի համար, ինչպես նաև կտոր առ կտոր բազմանդամներից բաղկացած հաջորդականությունների համար: Չիսելսկու համակարգով շարքերի համար որոշակի պայմանների առկայության դեպքում ստացվել է միակության թեորեմ:

Заключение

В диссертации рассмотрены вопросы базисности, сходимости и единственности для систем сплайнов.

В первой главе получены окончательные характеризации всех последовательностей, для которых соответствующая периодическая система Франклина является базисом или безусловным базисом в простраствах Лебега, Харди и в пространстве введеной Г. Соузой [23,24]. Несмотря на то, что последнее пространство является собственным подпространством пространства Харди, тем не менее периодическая система Франклина или является базисом (безусловным базисом) в обоих пространствах, или она не является базисом ни в одном из них. Получена также геометрическая характеризация всех последовательностей, для которых соответствующая ортонормальная система сплайнов высшего порядка является безусловным базисом в пространстве Харди. Как для периодической системы Франклина, так и для ортонормальной системы сплайнов высшего порядка при доказательстве безусловной базисности в пространстве Харди используется эквивалентность L^1 норм соответствующей квадратной функции Пэли и мажоранты частичных сумм. Мы доказали эквивалентность L^p норм по двоичным отрезкам квадратной функции Пэли и мажоранты частичных сумм для рядов по классической системе Франклина для всех положительных p при некотором условии. Также показано необходимость этого условия.

Во второй главе исследуется вопрос базисности систем сплайнов с нулевыми средними на оси в пространсте Харди. Получены необходимые и достаточные условия для порождающей последовательности при которых соответствующая система сплайнов с нулевыми средними является базисом в прострастве Харди. Полученные условия различны от соответствующих условий полученных для ортонормальных систем сплайнов определенных на единичном отрезке. Получены также необходимые и достаточные условия для безусловной базисности системы Франклина с нулевыми средними на оси в пространстве Харди.

Третья глава посвящена вопросам сходимости. Доказано, что ряд по общей системе Франклина безусловно сходится почти всюду на некотором множестве тогда и только тогда, когда этот ряд абсолютно сходится почти всюду на том же множестве. Также, построены системы сплайнов второго порядка такие, что ряд Фурье каждой непрерывной функции, которая не превышает наперед заданную функцию, локально равномерно сходится.

Четвертая глава посвящена вопросам единственности. Доказано, что адитивную функцию, определенную на P-адичных параллелепипедах, можно восстановить из ее производной, если мажоранта удовлетворяет некоторому условию. Применяя этот результат получены формулы восстанавливающие коэффициенты кратных рядов по системам Хаара, Уолша и их обобщений из суммы ряда, если мажоранта частичных сумм удовлетворяет некоторому условию. Похожие теоремы получены для рядов по классической системе Франклина, а также для последовательностей кусочно полиномиальных функций. При некотором условии также получена теорема единственности для рядов по системе Чисельского.