A BOUNDARY FOR THE EXISTENCE OF SOLUTION TO THE MAXIMUM ENTROPY PROBLEM APPLIED IN EUROPEAN CALL OPTIONS

N. D. MARGARYAN *

Department of Mathematical Modeling in Economics YSU, Armenia

The following paper introduces a computationally effective way of finding and utilizing maximum entropy problem boundaries for up to three dimensional cases. The application of the results is concentrated on financial options pricing and reverse distribution calculation. Based on market information in form of current option prices a distribution of future states is constructed. Using the suggested approach it will be possible to identify cases, where no solution to the maximum entropy problem exists, and parameters, for which a feasible solution can be reached.

MSC2010: Primary 52A15; Secondary 62P05, 91B25.

Keywords: boundary, entropy, distribution, options.

Introduction. The maximum entropy approach is being widely studied for its applications in different fields [1–4] including mathematical finance. The majority of such researches are done utilizing Shannon's information entropy [5] as their base starting point. Quite recently this approach has been applied in financial derivatives, more precisely, given a set of European call option prices a discrete distribution corresponding to a state maximum entropy can be calculated, resulting in market implied expectations that contradict the current market information the least [6]. Later, using Legendre transform, it has been shown that the above-mentioned distribution is unique in the family of risk-neutral measures that maximize the entropy [7]. The problem has been also considered without use of Lagrange multipliers, but analyzing it within the partially finite convex programming instead, still yielding consistent results [8]. Another variation was employing risk neutral moments as alternative constraints to option prices [9]. Maximum entropy concept has also been used as a non-parametric approach in pricing of American options [10]. We shall discuss the case of European call options and try to find bounds for option prices, so that a solution to the maximization problem exists. By theory, in the discrete case, the price of a European call option should be equal to the mathematical expectation of future pay-offs' discounted value, thus lying in their convex hull. In reality, actual market

---

* E-mail: narek_margaryan@outlook.com

prices may be biased from the theoretical ones [11] and lie out of the convex hull. We will address that issue and find a minimal value of a future price to be considered, so that a solution to the maximization problem exists, and so one can proceed to a maximum entropy calculating algorithm without worrying about whether or not it's going to be applied on an empty set, giving inconsistent results. The case of a three dimensional space is especially interesting, as in practice most of the options trading volume is concentrated in up to three different strike prices.

The Principle of Maximum Entropy. The maximum entropy principle seeks a distribution, which maximizes the entropy while satisfying predefined linear constraints. As it has been shown [2], if there is a non-empty set of vectors x such that

$$Ax = b, \tag{1}$$

where $b \in \mathbb{R}^n$, A is an operator of maximal rank (this is a necessary condition, as later the inverse of corresponding Hess matrix will be needed), then using Newton–Raphson's algorithm, be it modified or not, one can calculate a discrete distribution p, which satisfies:

$$Ap = b, \tag{2}$$

$$\sum_{i=1}^{n} p_i = 1, \quad p_i \geq 0, \tag{3}$$

$$S(p) = \sum_{i=1}^{n} p_i \ln(p_i) \quad \text{is maximal.} \tag{4}$$

In case of European call options, the matrix A represents future payouts and is of the following form:

$$\begin{bmatrix} (X_1 - K_1)^+ & (X_2 - K_1)^+ & \dots & (X_n - K_1)^+ \\ \vdots & \vdots & \ddots & \vdots \\ (X_1 - K_n)^+ & (X_2 - K_n)^+ & \dots & (X_n - K_n)^+ \end{bmatrix} \tag{5}$$

where by x^+ is denoted $\max(x, 0)$, K_i represents the i-th strike price, which corresponds to b_i, the future accumulated value of option's current price. The choice of vector $X = (X_1, \dots, X_n)$, i.e. future prices of the underlying asset, for which we wish to calculate probabilities, is arbitrary. However, nevertheless the fact that Newton–Raphson's method will seek the solution in case it exists, the choice of X heavily affects the existence of the solution. We will hereby address the problem of finding boundaries for the existence of solution and how to solve it.

Existence of the Solution. Assume we have denoted the strike prices in an ascending order, i.e. $K_1 < K_2 < \cdots < K_n$, also let's assume that n is fixed and is equal to 3 (the case for lower dimensions can be considered analogously). Notice that the matrix A has a quite convenient form like

$$\begin{bmatrix} 0 & a & b & \cdots \\ 0 & 0 & c & \cdots \\ 0 & 0 & 0 & \cdots \end{bmatrix}, \tag{6}$$

where $a, b, c, \dots$ are positive scalars. So for any column of A X_i which is smaller than K_1 will be a null. Proceeding columns that have a value of X_i between K_1 and K_2 will

have one positive coordinate, and so at each step of passing over the next K_i a new dimension will be gained. So one can deduce that the set of basis vectors is

$$\left\{ \begin{bmatrix} K_2 - K_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} K_3 - K_1 \\ K_3 - K_2 \\ 0 \end{bmatrix}, \begin{bmatrix} K_3 - K_1 + t \\ K_3 - K_2 + t \\ t \end{bmatrix} \right\}, \tag{7}$$

for arbitrary $t > 0$. The resulting convex hull of these vectors (together with the zero vector) is illustrated in Fig. 1. We denoted the vectors of the illustration below as followed: A $= (0, 0, 0)$, B $= (K_2 - K_1, 0, 0)$, C $= (K_3 - K_1, K_3 - K_2, 0)$, D $= (K_3 - K_1 + t, K_3 - K_2 + t, t)$, S $= (b_1, b_2, b_3)$.
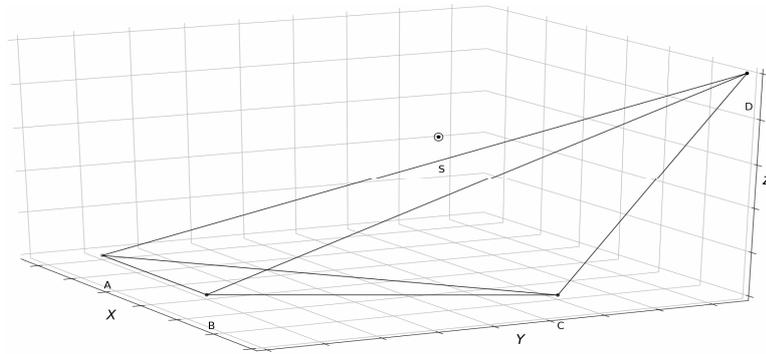

Fig. 1.

As we see, S may not be in the convex envelope of $\{A, B, C, D\}$. The only dynamic point in this system is D, it depends on the increment t, which corresponds to the maximum scalar of the vector X. By changing it, we can move the ABD plane "up" (where "up" direction means $(1, 0, 0)$, i.e. the Z-axis), until it crosses the point S. The ABD plane converges to its limit as $t \to \infty$.

P r o p o s i t i o n   1. The angle between vectors AD and AI, where I $= (1, 1, 1)$ converges to 0 as $t \to \infty$.

P r o o f . Let α denote the angle between unit vector AI and AD, then it's easy to show that

$$\lim_{t \to \infty} \cos \alpha(t) = \lim_{t \to \infty} \frac{\langle \text{AI}, \text{AD} \rangle}{\|\text{AI}\| \|\text{AD}\|} = 1. \tag{8}$$

The angle between AD and AI converges to 0, so S should be below ABI plane. On the other hand, S should also be above planes BCD and ACD. This conditions can be easily checked by calculating the normal of those planes and using the right hand thumb rule, resulting in the following. □

P r o p o s i t i o n   2 . Vector S will be in the convex hull of vectors $\{A, B, C, D\}$ if and only if the inequalities below hold:

$$\langle \text{CS}, [-1, -1, 0] \rangle \geq 0, \tag{9}$$

$$\langle \text{CS}, [K_3 - K_2, K_1 - K_3, K_2 - K_1] \rangle \geq 0, \tag{10}$$

$$\langle \text{AS}, [0, -1, 1] \rangle \leq 0. \tag{11}$$

Next, we wish to find t such that $S \in ABD$. To do this we note that the normal of plane ABD is of the form

$$[0, -t, K_3 - K_2 + t]. \tag{12}$$

Thus, t should be chosen to have AS orthogonal to the normal of ABD, resulting in the final statement.

P r o p o s i t i o n   3 . The minimal value of t for which a probability vector p satisfying conditions (2) and (3) exists is

$$t = \frac{b_3(K_3 - K_2)}{b_2 - b_3}. \tag{13}$$

Since the options with higher strikes have lower prices, $b_1 > b_2 > b_3$, t will be positive. The following two illustrations (Figs. 2 and 3) show the transition of plane ABD into ABD′, notice also that S belongs to ABD′.
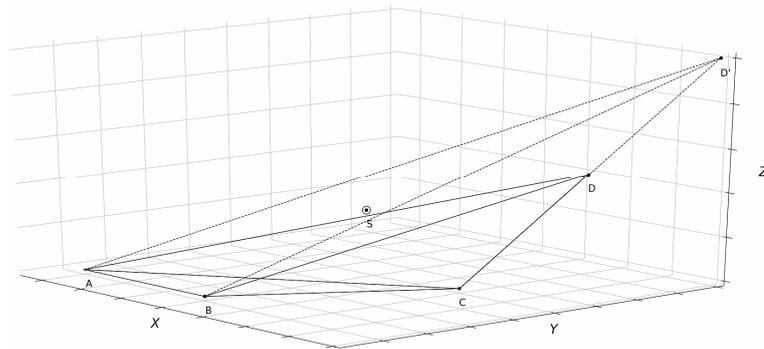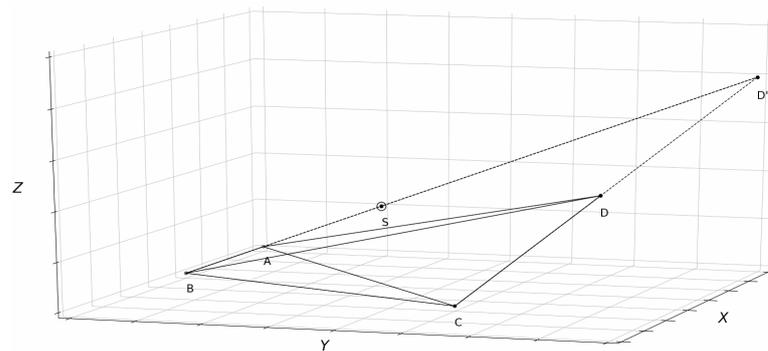

Fig. 2.

Fig. 3.

Conclusion. Let $X_i = K_i, i = 1 \ldots 3$, in the three dimensional form of matrix A given by (5) and let the fourth column of A be $[K_3 - K_1 + t, K_3 - K_2 + t, t]$ for some t. Then if a vector p satisfying (2) and (3) exists then the minimal value of t is given by (13). The above mentioned result can speed up the calculations, since no time will be wasted on maximizing entropy over an empty set or trying to randomly choose X such that a solution exists, providing a convenient tool for finding the

minimum necessary price needed and start entropy maximization from there. If one implements it to the maximum entropy distribution searching modified algorithm [12] it may automatically assign a minimum value to the price vector's last entry, so that a feasible solution will be found.

R E F E R E N C E S

1. **Alhassid Y.** et al. An Upper Bound for the Entropy and Its Applications to the Maximal Entropy Problem. // Chem. Phys. Lett., 1978, v. 53, p. 22.
2. **Alhassid Y.** et al. An Algorithm for Finding the Distribution of Maximal Entropy. // Journal of Computational Physics, 1979, v. 30, p. 250–258.
3. **Levine R.D., Tribus M.** The Maximum Entropy Formalism. MA, Cambridge: MIT Press, 1978, p. 207–209.
4. **Alhassid Y., Levine R.D.** Experimental and Inherent Uncertainties in the Information Theoretic Approach. // Chem. Phys. Lett., 1980, v. 73, p. 16–20.
5. **Shannon C.E.** A Mathematical Theory of Communication. // Bell Systems Technical Journal, 1948, v. 27, p. 379–423.
6. **Buchen P.W., Kelly M.** The Maximum Entropy Distribution of an Asset Inferred from Option Prices. // Journal of Financial and Quantitative Analysis, 1996, v. 31, p. 143–159.
7. **Neri C., Schneider L.** A Family of Maximum Entropy Densities Matching Call Options Prices. 2011.
   article Xiv:1102.0224v1 [q-fin.PR]
8. **Borwein J.** et al. Probability Distributions of Assets Inferred from Option Prices via the Principle of Maximum Entropy. // SIAM J. OPTIM., 2003, v. 14, № 2, p. 464–478.
9. **Rompolis L.S.** A New Method of Employing the Principle of Maximum Entropy to Retrieve the Risk Neutral Density. 2008.
10. **Xishen Yu, Li Yang** Pricing American Options Using a Nonparametric Entropy Approach. // Hindawi Publishing Corporation, 2014, 16 p.
    article ID 369795
11. **Rubinstein M.** Implied Binomial Trees. // Finance Working Paper, 1994, v. 49, № 3, p. 771–818.
12. **Margaryan N.D.** Assessment of Asset Price Distributions Using Maximum Entropy Method. // Proc. of Engineering Academy of Armenia, 2017, v. 14, № 1, p. 57–61.